# RECENT DEVELOPMENTS IN THE CALCULATION OF CMB ANISOTROPIES

Anthony Challinor and Anthony Lasenby
*MRAO, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, UK.*

February 1, 2008

**Abstract**

We discuss three recent developments in the calculation of Cosmic Microwave Background (CMB) anisotropies. We begin with a discussion of the relativistic corrections to the Sunyaev-Zel'dovich effect. By extending the Kompaneets equation to include relativistic effects, we are able to to derive simple analytic forms for the spectral changes due to the thermal Sunyaev-Zel'dovich effect. Relativistic corrections result in a small reduction in the amplitude of the effect over the Rayleigh-Jeans region, which for a typical cluster temperature of 8 keV, amounts to a correction downwards to the value of the Hubble constant derived from combined X-ray and Sunyaev-Zel'dovich information by about 5 percent. Our second topic is a discussion of covariant kinetic theory methods for the gauge-invariant calculation of primordial CMB anisotropies. We present a covariant version of the Boltzmann equation, which we use to derive a set of covariant equations for the gauge-invariant variables in a Cold Dark Matter (CDM) model, which are independent of the background curvature and type of perturbation. Equations describing a particular type of perturbation are obtained readily from this set, as we demonstrate for scalar perturbations in a $K = 0$ universe. By integrating the covariant Boltzmann equations along the line of sight, and using the instantaneous recombination approximation, we obtain an expression for the large scale anisotropy which agrees with an expression derived recently by Dunsby. Finally, we discuss some recent accurate calculations of the CMB anisotropy in global defect models. In such models, the effect of vorticity generation by the causal sources proves to be significant, leading to a suppression of acoustic peaks. The result is that global defect models of structure formation may already be at variance with the growing volume of CMB (and large scale structure) data.

# 1 Introduction

Future satellite experiments hold the promise of obtaining the first high resolution, full-sky maps of the CMB. These maps will contain a wealth of cosmological information, allowing unprecedented accuracy in the determination of cosmological parameters, and the opportunity to distinguish between rival theories of structure formation.

Interpretation of the CMB data requires not only accurate calculation of the primordial anisotropies in specific cosmological models, but also accurate modelling of sources of secondary anisotropies and foregrounds. In this lecture we consider three recent developments in the calculation of primary and secondary anisotropies. We begin, in Section 2, with secondary anisotropies, considering the effect of relativistic corrections on the spectral distortion of the CMB from Compton scattering in hot clusters [1] (the Sunyaev-Zel'dovich effect). We then turn to primary anisotropies, presenting a summary of some work in progress [2] on gauge-invariant calculations of the propagation of primordial anisotropies, using covariant kinetic theory methods, in Section 3. We use a version of the covariant perturbation theory developed by Ellis and coworkers (see, for example, Ellis *et al.* [3, 4]). Their approach is superior to others, including the Bardeen formalism [5], since it deals only with physically-defined, gauge-invariant variables. We end with a brief review of accurate calculations by Pen, Seljak and Turok [6, 7] of the CMB power spectra in global defect theories, in Section 4. These calculations show that there are problems reconciling global defect models of structure formation with observation.

We employ natural units $c = \hbar = G = 1$, except when presenting numerical results in Section 2.

# 2 Relativistic Corrections to the Sunyaev-Zel'dovich Effect

The Sunyaev-Zel'dovich effect is concerned with the generation of CMB spectral distortions by inverse Compton scattering in hot clusters. Non-relativistic treatments of this effect usually employ the Kompaneets equation [8] to determine the distortion. However, the Kompaneets equation does not include relativistic effects, which may be important for hot clusters where $k_B T_e \gtrsim 10\,\mathrm{keV}$. For this reason, and because of the low optical depth of typical clusters, relativistic treatments of the Sunyaev-Zel'dovich effect usually employ a multiple scattering description of the Comptonization process [9]–[13]. Including relativistic effects in this procedure gives a complicated expression for the spectral distortion, which is best handled by numerical techniques (see, for example, Rephaeli [13]).

In this section, we present an extension of the Kompaneets equation which includes relativistic effects in a self-consistent manner. The relativistic extension to the Kompaneets equation has been considered independently by Stebbins [14, 15]. The extended Kompaneets equation allows the Sunyaev-Zel'dovich effect in hot clusters to be described analytically on the basis of a Kompaneets type equation, instead of the previous numerical approaches. Simple analytic forms are given for the spectral distortions in the limit of small optical depth, including relativistic effects to second-order. These are in excellent agreement with Rephaeli's numerical calculations [13], which were based on the multiple scattering approach (truncated at one scattering).

This lends further support to Fabbri's observation that the Boltzmann equation can be applied to describe the Sunyaev-Zel'dovich effect in optically thin clusters [10], despite claims to the contrary [9].

The relativistic corrections to the Sunyaev-Zel'dovich effect are important in the calculation of the Hubble constant $H_0$ by the Sunyaev-Zel'dovich route in hot clusters (see, for example, Lasenby and Jones [16] for a recent review of the Sunyaev-Zel'dovich route of determining $H_0$, and Saunders [17] for recent observations). In the Rayleigh-Jeans region, we find that relativistic effects lead to a small decrease in the Sunyaev-Zel'dovich effect, and hence a small reduction in the hitherto determined values of $H_0$, in agreement with the conclusions of Rephaeli and Yankovitch [18].

## 2.1 Extending the Kompaneets Equation

In this lecture we shall not consider effects due to the peculiar motion of the cluster (such effects give rise to a kinetic correction to the Sunyaev-Zel'dovich effect). For a comoving cluster, the CMB photon distribution function is isotropic and may be denoted $n(\omega)$, where $\omega$ is the photon frequency. The electrons are assumed to be in thermal equilibrium at temperature $T_e$, and are described by an isotropic distribution function $f(E)$, where $E$ is the electron energy. The Boltzmann equation describing the evolution of $n(\omega)$ may be written as [19]

$$\frac{\partial n(\omega)}{\partial t} = -2 \int \frac{d^3p}{(2\pi)^3} d^3p' d^3k'\, W \Big[ n(\omega)\Big(1+n(\omega')\Big)f(E) - n(\omega')\Big(1+n(\omega)\Big)f(E') \Big], \tag{2.1}$$

where $W$ is the invariant transition amplitude for Compton scattering of a photon of 4-momentum $k^\mu$ by an electron (of charge $e$ and mass $m$) with 4-momentum $p^\mu$, to a photon momentum $k'^\mu$ and an electron momentum $p'^\mu$ [20]:

$$\begin{aligned} W &= \frac{(e^2/4\pi)^2 \bar{X}}{2\omega\omega' E E'} \delta^4(p+k-p'-k') \\ \bar{X} &\equiv 4m^4 \left(\frac{1}{\kappa} + \frac{1}{\kappa'}\right)^2 - 4m^2 \left(\frac{1}{\kappa} + \frac{1}{\kappa'}\right) - \left(\frac{\kappa}{\kappa'} + \frac{\kappa'}{\kappa}\right), \end{aligned} \tag{2.2}$$

with $\kappa \equiv -2p^\mu k_\mu$ and $\kappa' \equiv 2p^\mu k'_\mu$. In equation (2.1), we have assumed that electron degeneracy effects may be ignored.

The electrons are described by a relativistic Fermi distribution. Since we are ignoring degeneracy effects, we have

$$f(E) \approx e^{-(E-\mu)/k_B T_e}. \tag{2.3}$$

Substituting this form for $f(E)$ into equation (2.1), and expanding the term in brack-

ets in the integrand in powers of $\Delta x$, where

$$x \equiv \frac{\omega}{k_B T_e} \tag{2.4}$$

$$\Delta x \equiv \frac{\omega' - \omega}{k_B T_e}, \tag{2.5}$$

gives a Fokker-Planck expansion

$$\begin{aligned} \frac{\partial n(x)}{\partial t} = 2\left(\frac{\partial n}{\partial x} + n(1+n)\right) I_1 + 2\left(\frac{\partial^2 n}{\partial x^2} + 2(1+n)\frac{\partial n}{\partial x} + n(1+n)\right) I_2 \\ + 2\left(\frac{\partial^3 n}{\partial x^3} + 3(1+n)\frac{\partial^2 n}{\partial x^2} + 3(1+n)\frac{\partial n}{\partial x} + n(1+n)\right) I_3 \\ + 2\left(\frac{\partial^4 n}{\partial x^4} + 4(1+n)\frac{\partial^3 n}{\partial x^3} + 6(1+n)\frac{\partial^2 n}{\partial x^2} + 4(1+n)\frac{\partial n}{\partial x} + n(1+n)\right) I_4 + \cdots, \end{aligned} \tag{2.6}$$

where

$$I_n \equiv \frac{1}{n!}\int \frac{d^3 p}{(2\pi)^3} d^3 p' d^3 k'\, W f(E) (\Delta x)^n, \tag{2.7}$$

which does not depend on $n(\omega)$.

The calculation of the $I_n$ may be performed by expanding the integrand in powers of $p/m$ and $\omega/m$. Performing the integral over $E$ requires an asymptotic expansion of the electron distribution function $f(E)$. Evaluating the $I_n$, we develop a (possibly asymptotic) expansion of $\partial n/\partial t$ in $\theta_e$, where

$$\theta_e \equiv \frac{k_B T_e}{m}. \tag{2.8}$$

In this lecture, we shall only be concerned with the lowest order relativistic corrections, and start by retaining all terms up to $\mathrm{O}(\theta_e^2)$. To include all such terms consistently, it is necessary to retain only the first four terms in the series (2.6). A lengthy calculation gives the result

$$\frac{\partial n(x)}{\partial t} = \sigma_T N_e \theta_e \frac{1}{x^2}\frac{\partial}{\partial x}\Big(x^2 j(x)\Big), \tag{2.9}$$

where $N_e$ is the electron number density and the current $j(x)$ is given by

$$\begin{aligned} j(x) = x^2 \Bigg[ \Big(\frac{\partial n}{\partial x} + n(1+n)\Big) + \theta_e \Big[ \frac{5}{2}\Big(\frac{\partial n}{\partial x} + n(1+n)\Big) + \frac{21}{5} x \frac{\partial}{\partial x}\Big(\frac{\partial n}{\partial x} + n(1+n)\Big) \\ + \frac{7}{10} x^2 \Big(\frac{\partial^3 n}{\partial x^3} + 2\frac{\partial^2 n}{\partial x^2}(1+2n) + \frac{\partial n}{\partial x}\Big(1 - 2\frac{\partial n}{\partial x}\Big)\Big)\Big] + \mathrm{O}(\theta_e^2)\Bigg]. \end{aligned} \tag{2.10}$$

The zero-order term in equation (2.10) is just that term which usually appears in the Kompaneets equation [8]. The $\mathrm{O}(\theta_e)$ term is the lowest-order relativistic correction to the current. The form of equation (2.9) ensures conservation of the total number of photons, which is true for each order in $\theta_e$. A similar equation has been derived independently by Stebbins [15] using non-covariant methods, although he only includes terms to lowest-order in $\omega/m$ which is an excellent approximation for CMB photons.

## 2.2 The Sunyaev-Zel'dovich Effect

In this section we apply the generalised Kompaneets equation (to first-order in relativistic corrections) to the calculation of the Sunyaev-Zel'dovich effect in optically thin clusters. We consider higher-order effects in the next section.

Following the standard assumptions, we assume that the optical depth is sufficiently small that the spectral distortions are small. In this limit, we may solve equation (2.9) iteratively. The lowest order solution is obtained by substituting the initial photon distribution $n_0(x)$ into the current (eq. (2.10)). The integral over time is then trivial, and may be replaced by an integral along the line of sight through the cluster, giving

$$\Delta n(x) = \frac{y}{x^2}\frac{\partial}{\partial x}\Big(x^2 j(x)\Big), \tag{2.11}$$

where $j(x)$ is evaluated with $n_0(x)$, and

$$y \equiv \sigma_T \int N_e \theta_e \, dl, \tag{2.12}$$

where the integral is taken along the line of sight through the cluster.

For the CMB we take the initial (undistorted) photon distribution to be Planckian with temperature $T_0$:

$$n_0(x) = \frac{1}{e^{\alpha x} - 1}, \tag{2.13}$$

where $\alpha \equiv T_e/T_0$ is the (large) ratio of electron temperature to the CMB temperature. Evaluating equation (2.11) in the limit of large $\alpha$, we find the following fractional distortion:

$$\begin{aligned}\frac{\Delta n(X)}{n(X)} = \frac{yXe^X}{e^X - 1}\Big[X\coth(\tfrac{1}{2}X) - 4 + \theta_e\Big(-10 + \frac{47}{2}X\coth(\tfrac{1}{2}X) - \frac{42}{5}X^2\coth^2(\tfrac{1}{2}X) \\ + \frac{7}{10}X^3\coth^3(\tfrac{1}{2}X) + \frac{7X^2}{5\sinh^2(\tfrac{1}{2}X)}\Big(X\coth(\tfrac{1}{2}X) - 3\Big)\Big)\Big],\end{aligned} \tag{2.14}$$

correct to first-order in relativistic effects, where

$$X \equiv \frac{\hbar\omega}{k_B T_0}. \tag{2.15}$$

The first two terms in square brackets in equation (2.14) give the usual non-relativistic Sunyaev-Zel'dovich expression, while the terms proportional to $\theta_e$ are the lowest order relativistic correction. Equation (2.14) agrees with the result derived by Stebbins [15]. In the Rayleigh-Jeans limit (small $X$), we find

$$\frac{\Delta n(X)}{n(X)} \simeq -2y\left(1 - \frac{17}{10}\theta_e + \mathrm{O}(\theta_e^2)\right). \tag{2.16}$$

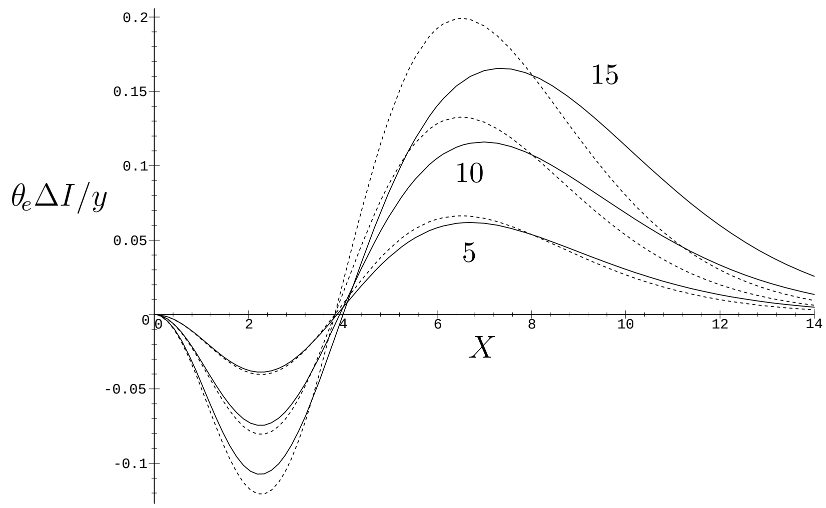


Figure 1: The intensity change $\theta_e \Delta I/y$ (in units of $2(k_B T_0)^3/(hc)^2$) plotted against $X$ for three values of $k_B T_e$ (in keV). The solid curves are calculated using the first-order correction to the Kompaneets equation, while the dashed lines are calculated from the usual Kompaneets expression.

In Figure 1 we plot $\theta_e \Delta I/y$ as a function of $X$, where

$$\Delta I = \frac{X^3}{e^X - 1} \frac{\Delta n}{n}, \tag{2.17}$$

is the change in spectral intensity in units of $2(k_B T_0)^3/(hc)^2$. Also plotted in Figure 1 are the non-relativistic predictions made with the standard Kompaneets equation. The curves in Figure 1 are for $k_B T_e = 5$, 10, and 15 keV, which are the same as the parameters used by Rephaeli [13] in his Figure 1. His calculations, which were based on the multiple scattering formalism [9] and required a numerical analysis, give results in excellent agreement with ours, which only require the use of the simple expression (2.7). This suggests that there is no difficulty in principle with applying the Boltzmann equation to the problem of Comptonization in clusters even though the optical depth may be very small. Similar conclusions were reached by Fabbri [10], but his demonstration was restricted to low temperature clusters where relativistic effects are not important.

It is clear from Figure 1 that for $X \lesssim 8$, the relativistic corrections lead to a reduction in the magnitude of the intensity change, compared to the non-relativistic prediction. The Hubble constant is inferred from combined Sunyaev-Zel'dovich and X-ray data by a relation of the form [16]

$$H_0 \propto \Delta I^{-2}, \tag{2.18}$$

where $\Delta I$ is the observed intensity change. The reduction in the magnitude of $\Delta I$ in the Rayleigh-Jeans region, for given cluster parameters, amounts to a decrease in the constant of proportionality in (2.18) and hence a reduction in the value of the Hubble constant that should be inferred.

## 2.3 Higher-order Effects

We have found that for $k_B T_e \gtrsim 10\,\mathrm{keV}$ the second-order relativistic effects make a significant contribution to the spectral distortion, while third-order effects are only significant for $k_B T_e \gtrsim 15\,\mathrm{keV}$.

These calculations require a straightforward extension of the method of Section 2.1 to include terms at $\mathrm{O}(\theta_e^3)$ (for second-order relativistic effects). For the calculation to $\mathrm{O}(\theta_e^3)$, it is necessary to retain the first six terms of the series (2.6), and to calculate $I_1$ through $I_6$ to $\mathrm{O}(\theta_e^3)$. The first iteration of equation (2.9) for $T_e \gg T_0$ gives the following next order (in $\theta_e$) correction to $\Delta n/n$:

$$\begin{aligned}\left(\frac{\Delta n(X)}{n(X)}\right)^{(2)} &= \theta_e^2 \frac{yXe^X}{e^X-1}\Big[-\frac{15}{2}+\frac{1023}{8}X\coth(\tfrac{1}{2}X)-\frac{868}{5}X^2\coth^2(\tfrac{1}{2}X)\\ &\quad+\frac{329}{5}X^3\coth^3(\tfrac{1}{2}X)-\frac{44}{5}X^4\coth^4(\tfrac{1}{2}X)+\frac{11}{30}X^5\coth^5(\tfrac{1}{2}X)\\ &+\frac{X^2}{30\sinh^2(\frac{1}{2}X)}\Big(-2604+3948X\coth(\tfrac{1}{2}X)-1452X^2\coth^2(\tfrac{1}{2}X)+143X^3\coth^3(\tfrac{1}{2}X)\Big)\\ &\quad+\frac{X^4}{60\sinh^4(\frac{1}{2}X)}\Big(-528+187X\coth(\tfrac{1}{2}X)\Big)\Big]. \end{aligned} \tag{2.19}$$

In the Rayleigh-Jeans limit, we find

$$\frac{\Delta n(X)}{n(X)} \simeq -2y\left(1-\frac{17}{10}\theta_e+\frac{123}{40}\theta_e^2+\mathrm{O}(\theta_e^3)\right). \tag{2.20}$$

In Figure 2 we compare the spectrum of $\Delta I$ calculated with equation (2.14) to the spectrum with the correction (2.19) included, for $k_B T_e = 5$, 10 and 15 keV ($\theta_e \approx 0.01$, 0.02 and 0.03 respectively). In each case, the second-order relativistic effects are not significant in the Rayleigh-Jeans part of the spectrum. This is to be expected from inspection of equation (2.20), where the $\theta_e^2$ term is clearly insignificant for the values of $\theta_e$ considered. For $k_B T_e = 5\,\mathrm{keV}$, the second-order effects are insignificant over the entire spectrum. However, for $k_B T_e \gtrsim 10\,\mathrm{keV}$, the second-order effects make a significant contribution to the relativistic correction to the Kompaneets-based prediction outside the Rayleigh-Jeans region. We have verified that the third-order corrections are negligible over the entire spectrum for $k_B T_e \simeq 10\,\mathrm{keV}$. This has been confirmed independently by a direct Monte-Carlo evaluation of the Boltzmann collision integral by Gull and Garrett [21]. The second-order effects should be included in the analysis of high frequency data for hot clusters. The magnitude of the second-order correction

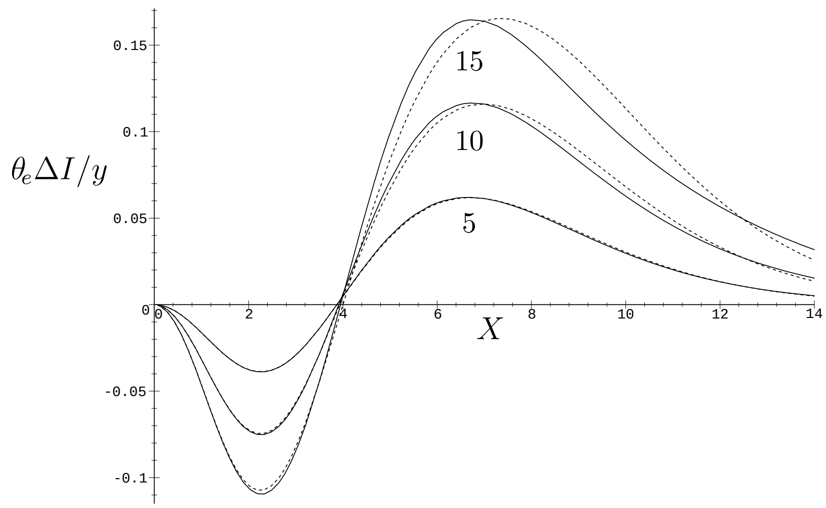


Figure 2: The intensity change $\theta_e \Delta I/y$ (in units of $2(k_B T_0)^3/(hc)^2$) plotted against $X$ for three values of $k_B T_e$ (in keV). The solid curves are calculated using the second-order correction to the Kompaneets equation, while the dashed lines are calculated from the first-order correction.

to the Sunyaev-Zel'dovich result for the rather mild values of $\theta_e$ considered here, is symptomatic of the asymptotic nature of the series expansion of $\partial n/\partial t$ in $\theta_e$. However, for the majority of clusters considered in Sunyaev-Zel'dovich analyses, the inclusion of the first two relativistic corrections is sufficient, particularly for experiments working in the Rayleigh-Jeans region of the spectrum.

## 2.4 The Crossover Frequency

The accurate determination of the crossover frequency $X_0$ (where the thermal component of the spectral distortion vanishes) is essential for reliable subtraction of the kinematic contribution to the Sunyaev-Zel'dovich effect [13]. In Figure 3 we plot the crossover frequency as a function of $k_B T_e$, with the first three relativistic corrections included. For $k_B T_e \lesssim 20\,\mathrm{keV}$ we find that $X_0$ is well approximated by the linear relation

$$X_0 \simeq 3.83(1 + 1.13\theta_e). \tag{2.21}$$

For comparison, Rephaeli [13], found $X_0$ to be approximated by $X_0 \simeq 3.83(1 + \theta_e)$ in the interval $k_B T_e = 1$–$50\,\mathrm{keV}$, while Fabbri [10] found $X_0 \simeq 3.83(1 + 1.1\theta_e)$ for $k_B T_e \lesssim 150\,\mathrm{keV}$. It is clear that our calculation favours Fabbri's expression. For $k_B T_e \gtrsim 20\,\mathrm{keV}$, $X_0$ calculated with the first three relativistic corrections departs from

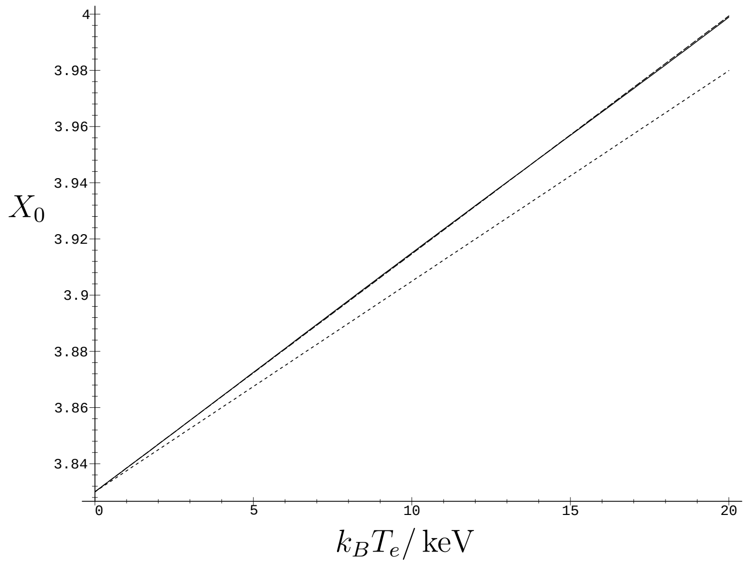


Figure 3: The crossover frequency $X_0$ plotted against $k_BT_e$. The solid line is calculated with the inclusion of third-order corrections to the Kompaneets equation. The upper dotted line is a linear fit to the solid line with $X_0 = 3.83(1 + 1.13\theta_e)$, while the lower dotted line is the linear fit given by Rephaeli: $X_0 = 3.83(1 + \theta_e)$.

the linear prediction (2.21). However, we do not regard this as indicative of a breakdown of the linear approximation, since it is clear from Figure 2 that the inclusion of higher-order terms may have a significant effect on the value of the crossover frequency.

## 2.5 Conclusion

The application of the relativistic extension of the Kompaneets equation, presented here, to the Sunyaev-Zel'dovich effect in optically thin clusters results in simple analytic expressions for the relativistic corrections to the usual Kompaneets based expression for the spectral distortion. These expressions are in excellent agreement with the numerical calculations of Rephaeli [13] for electron temperatures $\lesssim 10\,\mathrm{keV}$, providing further evidence that the low optical depth of clusters does not forbid the application of the Boltzmann equation to the calculation of the Sunyaev-Zel'dovich effect [10]. The asymptotic nature of the series expansion of $\partial n/\partial t$ in $\theta_e$ requires the inclusion of higher-order corrections to calculate the effect in hotter clusters in the Wien region of the spectrum. While the calculation of higher-order corrections is not problematic, the bad convergence properties of the series means that ultimately one must resort to a numerical calculation of the collision integral [21, 22] (or employ the multiple scattering formalism [9]) to calculate the distortion in very hot clusters.

Our calculations fully support the conclusions reached in Rephaeli [18], that in-

cluding relativistic effects leads to a small decrease in the value of the Hubble constant, inferred from combined X-ray and Sunyaev-Zel'dovich information. For a cluster temperature of $\simeq 8\,\mathrm{keV}$, the reduction in $H_0$ due to relativistic effects is $\simeq 5$ percent for measurements made in the Rayleigh-Jeans region.

# 3 Covariant and Gauge-Invariant Calculations of CMB Anisotropies

In this section we consider a covariant, kinetic theory approach for the gauge-invariant calculation of CMB anisotropies. We employ the elegant gauge-invariant perturbation methods of Ellis *et al.* [3, 4] (see also the recent Erice lectures by Ellis [23, 24] and the contribution to this volume from Ellis and Dunsby), although we prefer to express this within the gauge-theoretic approach to gravity (GTG) developed recently in Cambridge, instead of conventional general relativity. GTG has been discussed in previous Erice lectures [25, 26], although the most detailed discussion to date is contained in Lasenby, Doran and Gull [27]. In GTG, gravity is introduced via gauge-fields over a flat (background) spacetime. The gauge-theoretic purpose of the gauge-fields is to ensure covariance of the theory under smooth remappings of physical events onto vectors in the background spacetime, and under local rotations of geometric quantities at a point in the vector space. Physical observables must be extracted from the theory in a gauge-invariant manner, which ensures that the background vector space does not play an active role in the theory. GTG differs from general relativity only in certain circumstances, such as the treatment of horizons [27], solutions of general relativity with non-trivial topology, and the effects of quantum spin [28, 29]. In situations where none of these effects are present, such as the discussion of small departures from an exact Friedmann-Robertson-Walker (FRW) universe we consider here, the predictions of GTG coincide with general relativity. Formulating GTG in the powerful language of geometric algebra [30] allows GTG to be expressed in a coordinate-free manner, where all algebraic manipulations may be performed efficiently in the geometric algebra of spacetime [31] (the STA). Our notation and conventions for GTG are the same as in Lasenby, Doran and Gull [27] (see also the previous Erice lectures [25, 26]).

The covariant and gauge-invariant treatment of cosmological perturbations pioneered by Ellis *et al.* has many advantages over other approaches. The variables are covariantly-defined and relate to observable quantities in a simple manner. Furthermore, the variables are gauge-invariant, in that they are independent of the way that the 'lumpy' universe is mapped onto the background FRW universe (equivalently, a variable is gauge-invariant in this context if it transforms covariantly under gauge transformations of the lumpy universe, with the gauge for the background model held fixed). Note that in the covariant approach, the variables are truly gauge-invariant, which should be contrasted to the variables in Bardeen's [5] 'gauge-invariant' approach, which are gauge-invariant only if the scalar, vector or tensor nature of the

perturbations is not altered by the remapping. A further advantage of the covariant approach is that the perturbation equations may be formulated without the need for a scalar, vector or tensor splitting of the perturbations, or the associated harmonic analysis, whereas these are required at the outset in the Bardeen approach. The equations pertaining to a given type of perturbation may be obtained readily from the full perturbation equations by placing appropriate restrictions on the gauge-invariant variables.

The covariant calculation of large scale CMB anisotropies is dealt with in the lecture by Ellis [32], where a two fluid model of the matter and radiation are used, and a sharp last scattering surface is assumed. On small scales, the particle-like nature of the radiation (and neutrinos) becomes important, as well as the complicated local physics of recombination and decoupling. An accurate treatment of small scale effects requires a phase-space description of the photons, and the solution of the Boltzmann equation describing the evolution of the distribution function. Similarly, a phase-space description of the neutrinos is necessary to deal properly with the heat flux and anisotropic stress terms in the neutrino stress-energy tensor. In this section, we use covariant distribution functions which we decompose covariantly into angular moments, as discussed by Ellis *et al.* [23, 33, 34, 35]. In these papers, the discussion is limited to the decomposition of the Liouville equation, appropriate for freely propagating radiation. Here we present the covariant Boltzmann equation (for Thomson scattering of radiation off free electrons), and its angular moments. At present, polarisation is not included for simplicity. Since the generation of photon polarisation is sourced by anisotropy for Thompson scattering, the error resulting from the neglect of polarisation is small [36]. Using this formalism, we discuss the covariant equations describing the evolution of perturbations in a Cold Dark Matter (CDM) model. Our (linearised) equations are independent of scalar, vector or tensor decomposition, and of the spatial curvature of the background FRW model. Specialising to scalar perturbations in a $K = 0$ universe, we obtain a set of scalar mode equations for gauge-invariant variables. These equations must be solved numerically to obtain the present day anisotropies for given initial conditions. This work will be presented elsewhere [2]. We end this section with a discussion of large scale anisotropies, which is complementary to the discussion of Ellis and Dunsby [32]. The large scale anisotropy is obtained by integrating the covariant Boltzmann equation along null geodesics, and assuming instantaneous recombination. With the tight-coupling approximation, we obtain the same expression as that obtained by Ellis and Dunsby with a two-fluid model.

## 3.1 The Gauge-Invariant Variables

The starting point in the covariant description of cosmological perturbations is the choice of a velocity $u$. This should be chosen in a physical manner in such a way that if the universe is exactly FRW, the velocity reduces to that of the fundamental

observers. Having made a choice for $u$, we define a projection tensor $H(a)$ which projects vectors into the space perpendicular to $u$:

$$H(a) \equiv a - a\cdot uu. \tag{3.1}$$

For details of our geometric algebra notation see Lasenby *et al.* [25]–[27]. Expressing the theory in geometric algebra provides frame-free and coordinate-free versions of all variables and operations. Using the projection (3.1), we form a directional 'spatial' derivative, $a\cdot\widehat{\mathcal{D}}$ from the usual directional (covariant) derivative $a\cdot\mathcal{D}$ via

$$a\cdot\widehat{\mathcal{D}}M \equiv H[H(a)\cdot\mathcal{D}M], \tag{3.2}$$

where $M$ is an arbitrary multivector. We also define a derivative $a\cdot\bar{\mathcal{D}}$ which acts on tensors $T(b,\ldots,c)$ according to

$$a\cdot\bar{\mathcal{D}}T(b,\ldots,c) \equiv H[H(a)\cdot\dot{\mathcal{D}}\dot{T}(H(b),\ldots,H(c))], \tag{3.3}$$

where $a\cdot\dot{\mathcal{D}}\dot{T}(b,\ldots,c) = a\cdot\mathcal{D}T(b,\ldots,c) - T(a\cdot\mathcal{D}b,\ldots,c) - T(b,\ldots,a\cdot\mathcal{D}c)$. In an exact FRW universe, spatial gradients of any field $M$ necessarily vanish by homogeneity, so that $a\cdot\widehat{\mathcal{D}}M = 0$. It follows that the spatial gradient of the density $\widehat{\mathcal{D}}\rho$ is a gauge-invariant variable. Quantities such as $\widehat{\mathcal{D}}\rho$ (or, more correctly, the dimensionless quantity $\widehat{\mathcal{D}}\rho/(\rho H)$, where $H$ is a suitable average of the local Hubble parameter) will be small compared to unity if the universe is sufficiently close to an FRW universe. We shall refer to such variables as being first-order, or O(1). In the linear theory used here, products of first-order quantities may be neglected in an expression, compared to any first-order (or zero-order) quantities. It is convenient to introduce a scale factor $S$, satisfying

$$u\cdot\mathcal{D}S = HS, \qquad \widehat{\mathcal{D}}S = \mathrm{O}(1), \tag{3.4}$$

so that we may remove the variation in $\widehat{\mathcal{D}}\rho$ along an integral curve of $u$ (a 'worldline') by defining the variable

$$\mathcal{X} \equiv \frac{S\widehat{\mathcal{D}}\rho}{\rho}. \tag{3.5}$$

The first-order, gauge-invariant variable $\mathcal{X}$ is an example of a fractional, comoving spatial gradient.

The covariant derivative of $u$ decomposes as

$$a\cdot\mathcal{D}u = \tfrac{1}{2}a\cdot\varpi + \sigma(a) + \tfrac{1}{3}H(a)\theta + a\cdot uw, \tag{3.6}$$

where the vector $w \equiv u\cdot\mathcal{D}u$ is the acceleration, the bivector $\varpi \equiv \mathcal{D}\wedge u + w\wedge u$ is the vorticity, measuring the local rotation of the worldlines, and the scalar $\theta \equiv \mathcal{D}\cdot u = 3H$ measures the volume expansion rate. The traceless symmetric tensor $\sigma(a)$ measures

the shear of the worldlines, and is orthogonal to $u$: $\sigma(u) = 0$. We shall refer to tensors orthogonal to $u$ as being spatial tensors. The vorticity, acceleration and shear are all first-order gauge-invariant variables (they vanish in an FRW universe). However, like $\rho$, the volume expansion rate $\theta$ is zero-order, but we obtain a first-order gauge-invariant variable $\mathcal{Z}$ by taking the spatial gradient, $\mathcal{Z} \equiv S\widehat{\mathcal{D}}\theta$.

The matter stress-energy tensor $\mathcal{T}(a)$ decomposes as

$$\mathcal{T}(a) = \rho a{\cdot}uu + qa{\cdot}u + uq{\cdot}a - pH(a) + \pi(a), \tag{3.7}$$

where the scalar $\rho \equiv \mathcal{T}(u)\cdot u$ is the density of matter (measured by a comoving observer), the vector $q \equiv H[\mathcal{T}(u)]$ is the energy (or heat) flux, the scalar $p \equiv -H(\partial_a)\cdot \mathcal{T}(a)/3$ is the isotropic pressure, and the traceless, symmetric, spatial tensor $\pi(a) \equiv H[\mathcal{T}H(a)] + pH(a)$ is the anisotropic stress. In an exact FRW universe, the stress-energy tensor is forced to take the ideal fluid form, hence $q$ and $\pi(a)$ are first-order variables. A first-order, gauge-invariant variable may be derived from the pressure by taking a spatial gradient.

The final gauge-invariant variables we shall make use of derive from the Weyl tensor $\mathcal{W}(B)$, which is a bivector-valued linear function of a bivector $B$. It is convenient to split the Weyl tensor into electric and magnetic parts, in analogy with the split of the Faraday bivector in electromagnetism. We define the electric part by

$$\mathcal{E}(a) \equiv -u{\cdot}\mathcal{W}(a{\wedge}u), \tag{3.8}$$

and the magnetic part by

$$\mathcal{B}(a) \equiv -iu{\wedge}\mathcal{W}(a{\wedge}u), \tag{3.9}$$

where $i$ is the unit pseudoscalar for spacetime (which performs the duality operation in the STA). Both $\mathcal{E}(a)$ and $\mathcal{B}(a)$ are traceless, symmetric spatial tensors. The definitions (3.8) and (3.9) combine together to give the neat relation

$$\mathcal{W}(a{\wedge}u)u = \mathcal{E}(a) + i\mathcal{B}(a). \tag{3.10}$$

Since the Weyl tensor vanishes in an FRW universe, the electric and magnetic parts of the Weyl tensor are first-order gauge-invariant variables. The equations of motion for the gauge-invariant variables may be derived from the Ricci identity, the Bianchi identity, and the contracted Bianchi identity. These equations are the GTG equivalents of the equations given by Ellis [32] in his lecture, and will be given in linearised from in Section 3.4. The equations split into propagation equations (those involving the derivative along a worldline $u\cdot\mathcal{D}$), and constraint equations which involve only spatial derivatives. The constraint equations restrict the specification of initial data on a hypersurface (the data must satisfy the constraints), while the propagation equations allow one to propagate the data off the surface.

## 3.2 The Boltzmann Equation

The photon distribution function is a scalar valued function of position $x$ and covariant momentum $p$, denoted by $f_\gamma(x,p)$. Physically, an observer sees $f_\gamma(x,p)dV|d^3p|$ photons in a volume $dV$ and a 3-momentum volume $|d^3p|$. It is convenient to split the photon momentum $p$ as

$$p = E(U+e), \tag{3.11}$$

where $E \equiv pu$ is the photon energy and $e$ is a unit spacelike vector $e^2 = -1$ orthogonal to $u$. The photon distribution function may then be written as $f_\gamma(x,E,e)$. We will usually leave the $x$ dependence implicit, writing $f_\gamma(E,e)$. The stress-energy tensor for the photons may be written as

$$\mathcal{T}_\gamma(a) = \int dE d\Omega\, E f_\gamma(E,e) p\, p\cdot a, \tag{3.12}$$

where $d\Omega$ denotes an integral over solid angles.

Following Ellis *et al.* [35], we describe the angular dependence of $f_\gamma(E,e)$ by expanding in covariant, scalar-valued tensors $F_\gamma^{(l)}(a_1,a_2,\ldots,a_l)$:

$$f_\gamma(E,e) = \sum_{l=0}^{\infty} F_\gamma^{(l)}(e,e,\ldots,e). \tag{3.13}$$

The tensors $F_\gamma^{(l)}(a_1,a_2,\ldots,a_l)$ have an (implicit) dependence on $x$ and $E$ and satisfy the properties:

$$\begin{aligned} F_\gamma^{(l)}(a_1,\ldots,a_j,\ldots,a_k,\ldots,a_l) &= F_\gamma^{(l)}(a_1,\ldots,a_k,\ldots,a_j,\ldots,a_l) \\ F_\gamma^{(l)}(u,a_2,\ldots,a_l) &= 0 \\ F_\gamma^{(l)}(\partial_b,b,a_3,\ldots,a_l) &= 0. \end{aligned} \tag{3.14}$$

The action of the Liouville operator $\mathcal{L}$ on $F_\gamma^{(l)}(e,\ldots,e)$, evaluates to

$$\mathcal{L}F_\gamma^{(l)}(e,\ldots,e) = \partial_E F_\gamma^{(l)}(e,\ldots,e)\partial_\lambda E + p\cdot\dot{\mathcal{D}}\dot{F}_\gamma^{(l)}(e,\ldots,e) + lF_\gamma^{(l)}(p\cdot\mathcal{D}e,e,\ldots,e), \tag{3.15}$$

where $\lambda$ is the affine parameter to the photon path ($p = \underline{h}^{-1}(\partial_\lambda x)$ with $\bar{h}(a)$ the GTG position-gauge field). It follows that the action of the Liouville operator on $f_\gamma(E,e)$ may be written in the form

$$\mathcal{L}f_\gamma(E,e) = \sum_{l=0}^{\infty}[\partial_E F_\gamma^{(l)}(e,\ldots,e)\partial_\lambda E + p\cdot\dot{\mathcal{D}}\dot{F}_\gamma^{(l)}(e,\ldots,e) + lF_\gamma^{(l)}(p\cdot\mathcal{D}e,e,\ldots,e)]. \tag{3.16}$$

For $l>0$ the tensors $F_\gamma^{(l)}(a_1,\ldots,a_l)$ are first-order quantities. However, to zero-order, we have

$$p\cdot\mathcal{D}e = \tfrac{1}{3}\theta E u, \tag{3.17}$$

so that the final term in (3.16) only contributes at second-order, and may be ignored in the linearised calculation considered here.

Since neutrinos are essentially collisionless over the period of interest, their distribution function $f_\nu(x,p)$ evolves according to the Liouville equation $\mathcal{L}f_\nu(x,p)=0$. The photons, however, are in close contact with the baryons through Thomson scattering off free electrons (which are at rest in the frame of the baryons because of Coulomb interactions). In the presence of collisions, the photon distribution function evolves according to the Boltzmann equation

$$\mathcal{L}f_\gamma(x,p)=C, \tag{3.18}$$

where the collision term approximates to

$$C=n_e\sigma_T p\cdot u_b(f_+(x,p)-f(x,p)), \tag{3.19}$$

with $u_b$ denoting the baryon velocity, $\sigma_T$ the Thomson cross section, and $f_+(x,p)$ describes scattering into the phase space element under consideration. Evaluating $f_+(x,p)$ to first-order, multiplying by $E^2$ and integrating over energy, we find [2]

$$\begin{aligned}\int_0^\infty dE\, E^2\mathcal{L}f_\gamma(E,e) = \frac{3}{16\pi}\sigma_T n_e\left(\frac{4}{3}(1-4e\cdot v_b)\rho_\gamma + e\cdot\pi_\gamma(e)\right)&\\ -\sigma_T n_e\int_0^\infty dE\, E^3 f_\gamma(E,e),&\end{aligned} \tag{3.20}$$

where $v_b\equiv u_b-u$ is the first-order relative velocity of the baryons.

Equation (3.20) may itself be expanded in symmetric, trace-free tensors orthogonal to $u$ (like $F_\gamma^{(l)}(a_1,\ldots,a_l)$). The $l=0$, 1 and 2 terms require a little care in the linearised calculation since $F_\gamma^{(0)}$ is a zero-order quantity. The result is [2]:
for $l=0$

$$u\cdot\mathcal{D}\rho_\gamma+\tfrac{4}{3}\theta\rho_\gamma+\widehat{\mathcal{D}}\cdot q_\gamma=0, \tag{3.21}$$

for $l=1$

$$u\cdot\mathcal{D}q_\gamma+\tfrac{4}{3}\theta q_\gamma+\pi_\gamma(\bar{\mathcal{D}})+\tfrac{4}{3}w\rho_\gamma-\tfrac{1}{3}\widehat{\mathcal{D}}\rho_\gamma=\sigma_T n_e\left(\tfrac{4}{3}\rho_\gamma v_b-q_\gamma\right), \tag{3.22}$$

for $l=2$

$$\begin{aligned}a_1\cdot(u\cdot\dot{\mathcal{D}}\dot{\pi}_\gamma(a_2))+\tfrac{4}{3}\theta a_1\cdot\pi_\gamma(a_2)+J_\gamma^{(3)}(\bar{\mathcal{D}},a_1,a_2)-\tfrac{1}{5}\Big(a_1\cdot(a_2\cdot\widehat{\mathcal{D}}q_\gamma)+a_2\cdot(a_1\cdot\widehat{\mathcal{D}}q_\gamma)&\\ -\tfrac{2}{3}H(a_1)\cdot H(a_2)\widehat{\mathcal{D}}\cdot q_\gamma\Big)-\tfrac{8}{15}a_1\cdot\sigma(a_2)\rho_\gamma=-\tfrac{9}{10}\sigma_T n_e a_1\cdot\pi_\gamma(a_2),&\end{aligned} \tag{3.23}$$

and for $l\geq 3$

$$\begin{aligned}&u\cdot\dot{\mathcal{D}}\dot{J}_\gamma^{(l)}(a_1,\ldots,a_l)+\tfrac{4}{3}\theta J_\gamma^{(l)}(a_1,\ldots,a_l)+J_\gamma^{(l+1)}(\bar{\mathcal{D}},a_1,\ldots,a_l)\\ &-\tfrac{l}{(2l+1)l!}\Big(a_1\cdot\bar{\mathcal{D}}J_\gamma^{(l-1)}(a_2,\ldots,a_l)-\tfrac{(l-1)}{(2l-1)}J_\gamma^{(l-1)}(\bar{\mathcal{D}},a_1,\ldots,a_{l-2})H(a_{l-1})\cdot H(a_l)+\text{perms}\Big)\\ &\qquad=-\sigma_T n_e J_\gamma^{(l)}(a_1,\ldots,a_l).\end{aligned} \tag{3.24}$$

The scalar-valued, traceless, symmetric, spatial tensor $J_\gamma^{(l)}(a_1,\ldots,a_l)$ is defined by

$$J_\gamma^{(l)}(a_1,\ldots,a_l) \equiv \frac{4\pi(-2)^l(l!)^2}{(2l+1)(2l)!}\int_0^\infty dE\,E^3 F_\gamma^{(l)}(a_1,\ldots,a_l). \tag{3.25}$$

With this definition, we have

$$\rho_\gamma = J_\gamma^{(0)} \tag{3.26}$$

$$q_\gamma = \partial_a J_\gamma^{(1)}(a) \tag{3.27}$$

$$\pi_\gamma(a) = \partial_b J_\gamma^{(2)}(b,a). \tag{3.28}$$

It is not hard to show that the combination

$$a_1\cdot\bar{\mathcal{D}}J_\gamma^{(l-1)}(a_2,\ldots,a_l) - \tfrac{(l-1)}{(2l-1)}J_\gamma^{(l-1)}(\bar{\mathcal{D}},a_1,\ldots,a_{l-2})H(a_{l-1})\cdot H(a_l) + \text{perms}, \tag{3.29}$$

is a trace-free, symmetric tensor, orthogonal to $u$, as required. Note that the linearised equations show a coupling between the $l-1$, $l$ and $l+1$ terms, whereas the exact equations also show coupling between the $l-2$ and $l+2$ terms. The terms involving $J_\gamma^{(l-2)}(a_1,\ldots,a_{l-2})$ and $J_\gamma^{(l+2)}(a_1,\ldots,a_{l+2})$ also involve the shear tensor $\sigma(a)$. Ellis has remarked [24] that the exact result that if the hierarchy of $J_\gamma^{(l)}$ tensors truncates after a finite number of terms then the shear must vanish, is missed in the linearised approach. This is certainly true, however it is not problematic for the calculation of CMB anisotropies, since it is never claimed that the expansion of the photon distribution function truncates. Instead, the series is truncated (with suitable care to avoid reflection of power) after a finite number of terms for numerical convenience (see, for example, Ma and Bertshinger [37]). The truncation is chosen to be high enough up the series that it has negligible effect on the $J_\gamma^{(l)}$ for the range of $l$ of interest.

## 3.3 Baryons and Dark Matter

Over the epoch of interest here, the baryons and electrons are non-relativistic, and we assume that they may be described as an ideal fluid, with energy density $\rho_b$ (measured in the rest frame of the baryons), pressure $p_b$, and covariant-velocity $u+v_b$, where $v_b$ is a first-order quantity. The linearised baryonic stress-energy tensor evaluates to

$$\mathcal{T}_b(a) = \rho_b a\cdot uu - p_b H(a) + (\rho_b+p_b)(a\cdot uv_b + a\cdot v_b u), \tag{3.30}$$

which shows that there is a heat-flux $(\rho_b+p_b)v_b$ measured by an observer moving with velocity $u$. To find the equations of motion for $\rho_b$ and $v_b$, we make use of the fact that the baryons (including electrons) and photons interact non-gravitationally only with themselves, so that

$$\dot{\mathcal{T}}_\gamma(\dot{\mathcal{D}}) + \dot{\mathcal{T}}_b(\dot{\mathcal{D}}) = 0. \tag{3.31}$$

Using the moment equations of the previous section, we find a propagation equation for $\rho_b$:

$$u\cdot\mathcal{D}\rho_b + (\rho_b + p_b)\theta + (\rho_b + p_b)\widehat{\mathcal{D}}\cdot v_b = 0, \tag{3.32}$$

and a propagation equation for $v_b$:

$$(\rho_b + p_b)(u\cdot\mathcal{D}v_b + w) + \tfrac{1}{3}(\rho_b + p_b)\theta v_b + u\cdot\mathcal{D}p_b - \widehat{\mathcal{D}}p_b + \sigma_T n_e\left(\tfrac{4}{3}\rho_\gamma v_b - q_\gamma\right) = 0, \tag{3.33}$$

which must be supplemented by an equation of state linking $p_b$ and $\rho_b$. Note that equation (3.32) implies that there is no energy exchange between the radiation and the baryon plasma in the frame in which the baryon plasma is at rest, while equation (3.33) shows that there is momentum exchange due to dipole anisotropy of the CMB in the frame of the plasma. This behaviour is to be expected since we have assumed that the baryon-photon interaction approximates to Thomson scattering in the baryon rest frame, for which there will be no energy transfer, but there will be momentum transfer if the photon distribution function is not isotropic.

We will only consider cold dark matter (CDM) here, which may be described as a pressureless ideal fluid. Hot dark matter (HDM) would require a distribution function description which for massive particles greatly increases the numerical complexity of the problem. (Both CDM and HDM are considered, for example, in the gauge-dependent treatment in Ma and Bertshinger [37]). The CDM has energy density $\rho_c$ in its rest frame, which has covariant velocity $u + v_c$, with $v_c$ a first-order quantity. The CDM interacts with other species through gravity alone, so the equations of motion are

$$u\cdot\mathcal{D}\rho_c + \rho_c\theta + \rho_c\widehat{\mathcal{D}}\cdot v_c = 0 \tag{3.34}$$

$$u\cdot\mathcal{D}v_c + \tfrac{1}{3}\theta v_c + w = 0. \tag{3.35}$$

It is very convenient to use the CDM velocity to define the fundamental velocity $u$ ($v_c = 0$). Since the CDM is pressureless, it moves geodesically so that the acceleration $w$ will vanish for this choice of $u$. The discussion in the next two sections assumes that this choice has been made.

## 3.4 General Equations for the Evolution of CMB Anisotropies

As Ellis and Bruni [3] have shown, the natural variables with which to describe cosmological perturbations are the comoving fractional spatial gradients of the energy densities, $\mathcal{X}_i$ where $i$ labels the particle species. The equations of motion for these variables follow from taking the spatial gradients of the evolution equations for the

$\rho_i$. This procedure yields the linearised equations:

$$u\cdot\mathcal{D}\mathcal{X}_\nu = -\frac{4}{3}\mathcal{Z} - \frac{S}{\rho_\nu}\widehat{\mathcal{D}}\widehat{\mathcal{D}}\cdot q_\nu \tag{3.36}$$

$$u\cdot\mathcal{D}\mathcal{X}_\gamma = -\frac{4}{3}\mathcal{Z} - \frac{S}{\rho_\gamma}\widehat{\mathcal{D}}\widehat{\mathcal{D}}\cdot q_\gamma \tag{3.37}$$

$$u\cdot\mathcal{D}\mathcal{X}_c = -\mathcal{Z} \tag{3.38}$$

$$u\cdot\mathcal{D}\mathcal{X}_b = -\left(1+\frac{p_b}{\rho_b}\right)(\mathcal{Z} + S\widehat{\mathcal{D}}\widehat{\mathcal{D}}\cdot v_b) - S\theta\frac{\widehat{\mathcal{D}}p_b}{\rho_b} + \theta\mathcal{X}_b\frac{p_b}{\rho_b}. \tag{3.39}$$

An equation of motion for $\mathcal{Z}$ may be obtained by taking the spatial gradient of the propagation equation for $\theta$ (the Raychaudhuri equation, which follows from the Ricci identity) to obtain

$$u\cdot\mathcal{D}\mathcal{Z} = -\tfrac{2}{3}\theta\mathcal{Z} - \kappa\left(\rho_\gamma\mathcal{X}_\gamma + \rho_\nu\mathcal{X}_\nu + \tfrac{1}{2}\rho_c\mathcal{X}_c + \tfrac{1}{2}\rho_b\mathcal{X}_b + \tfrac{3}{2}S\widehat{\mathcal{D}}p_b\right). \tag{3.40}$$

Note that these equations do not close due to the presence of $v_b$, $q_\gamma$ and $q_\nu$. These equations, along with the kinetic theory equations given in the previous section and the equations for the gauge-invariant variables, discussed in Section 3.1, form a complete description of the evolution of CMB anisotropies. For convenience, we group these equations together in this section.

The baryon peculiar velocity $v_b$ evolves according to

$$(\rho_b+p_b)u\cdot\mathcal{D}v_b + \tfrac{1}{3}(\rho_b+p_b)\theta v_b + v_b u\cdot\mathcal{D}p_b - \widehat{\mathcal{D}}p_b + \sigma_T n_e\left(\tfrac{4}{3}\rho_\gamma v_b - q_\gamma\right) = 0. \tag{3.41}$$

For the neutrinos, we have the moment equations

$$u\cdot\mathcal{D}q_\nu + \tfrac{4}{3}\theta q_\nu + \pi_\nu(\bar{\mathcal{D}}) - \tfrac{1}{3}\widehat{\mathcal{D}}\rho_\nu = 0, \tag{3.42}$$

$$\begin{aligned} a_1\cdot(u\cdot\dot{\mathcal{D}}\dot{\pi}_\nu(a_2)) + \tfrac{4}{3}\theta a_1\cdot\pi_\nu(a_2) + J_\nu^{(3)}(\bar{\mathcal{D}},a_1,a_2) - \tfrac{1}{5}\Big(a_1\cdot(a_2\cdot\widehat{\mathcal{D}}q_\nu) + a_2\cdot(a_1\cdot\widehat{\mathcal{D}}q_\nu) \\ - \tfrac{2}{3}H(a_1)\cdot H(a_2)\widehat{\mathcal{D}}\cdot q_\nu\Big) - \tfrac{8}{15}a_1\cdot\sigma(a_2)\rho_\nu = 0, \end{aligned} \tag{3.43}$$

and for $l \geq 3$

$$\begin{aligned} &u\cdot\dot{\mathcal{D}}\dot{J}_\nu^{(l)}(a_1,\ldots,a_l) + \tfrac{4}{3}\theta J_\nu^{(l)}(a_1,\ldots,a_l) + J_\nu^{(l+1)}(\bar{\mathcal{D}},a_1,\ldots,a_l) \\ &- \tfrac{l}{(2l+1)l!}\Big(a_1\cdot\bar{\mathcal{D}}J_\nu^{(l-1)}(a_2,\ldots,a_l) - \tfrac{(l-1)}{(2l-1)}J_\nu^{(l-1)}(\bar{\mathcal{D}},a_1,\ldots,a_{l-2})H(a_{l-1})\cdot H(a_l) + \text{perms}\Big) \\ &= 0. \end{aligned} \tag{3.44}$$

For the photons, we have

$$u\cdot\mathcal{D}q_\gamma + \tfrac{4}{3}\theta q_\gamma + \pi_\gamma(\bar{\mathcal{D}}) - \tfrac{1}{3}\widehat{\mathcal{D}}\rho_\gamma = \sigma_T n_e\left(\tfrac{4}{3}\rho_\gamma v_b - q_\gamma\right), \tag{3.45}$$

$$a_1\cdot(u\cdot\dot{\mathcal{D}}\dot{\pi}_\gamma(a_2)) + \tfrac{4}{3}\theta a_1\cdot\pi_\gamma(a_2) + J_\gamma^{(3)}(\bar{\mathcal{D}}, a_1, a_2) - \tfrac{1}{5}\Big(a_1\cdot(a_2\cdot\widehat{\mathcal{D}}q_\gamma) + a_2\cdot(a_1\cdot\widehat{\mathcal{D}}q_\gamma) - \tfrac{2}{3}H(a_1)\cdot H(a_2)\widehat{\mathcal{D}}\cdot q_\gamma\Big) - \tfrac{8}{15}a_1\cdot\sigma(a_2)\rho_\gamma = -\tfrac{9}{10}\sigma_T n_e a_1\cdot\pi_\gamma(a_2), \quad (3.46)$$

and for $l \geq 3$

$$u\cdot\dot{\mathcal{D}}\dot{J}_\gamma^{(l)}(a_1,\dots,a_l) + \tfrac{4}{3}\theta J_\gamma^{(l)}(a_1,\dots,a_l) + J_\gamma^{(l+1)}(\bar{\mathcal{D}}, a_1,\dots,a_l) - \tfrac{l}{(2l+1)l!}\Big(a_1\cdot\bar{\mathcal{D}}J_\gamma^{(l-1)}(a_2,\dots,a_l) - \tfrac{(l-1)}{(2l-1)}J_\gamma^{(l-1)}(\bar{\mathcal{D}}, a_1,\dots,a_{l-2})H(a_{l-1})\cdot H(a_l) + \text{perms}\Big) = -\sigma_T n_e J_\gamma^{(l)}(a_1,\dots,a_l). \quad (3.47)$$

The remaining equations are the propagation and constraint equations for the kinematic variables and the electric and magnetic parts of the Weyl tensor. The constraint equations are

$$\mathcal{B}(a) - \tfrac{1}{4}[(ia\wedge u\wedge\widehat{\mathcal{D}})\cdot\varpi + iu\wedge\bar{\mathcal{D}}\wedge(a\cdot\varpi)] + \tfrac{1}{2}[iu\wedge\bar{\mathcal{D}}\wedge\sigma(a) + \sigma(iu\wedge a\wedge\bar{\mathcal{D}})] = 0 \quad (3.48)$$

$$\mathcal{B}(\bar{\mathcal{D}}) = -\tfrac{1}{2}\kappa[(\rho+p)iu\wedge\varpi + iu\wedge\widehat{\mathcal{D}}\wedge q] \quad (3.49)$$

$$\mathcal{E}(\bar{\mathcal{D}}) = \tfrac{1}{6}\kappa[2\widehat{\mathcal{D}}\rho + 2\theta q + 3\pi(\bar{\mathcal{D}})] \quad (3.50)$$

$$\tfrac{1}{2}\widehat{\mathcal{D}}\cdot\varpi - \sigma(\bar{\mathcal{D}}) + \tfrac{2}{3}\widehat{\mathcal{D}}\theta = -\kappa q \quad (3.51)$$

$$\widehat{\mathcal{D}}\cdot(iu\wedge\varpi) = 0, \quad (3.52)$$

where $\rho$ and $p$ are the total energy density and pressure measured by an observer moving with velocity $u$:

$$\rho \equiv \rho_\nu + \rho_\gamma + \rho_b + \rho_c \quad (3.53)$$

$$p \equiv \tfrac{1}{3}\rho_\nu + \tfrac{1}{3}\rho_\gamma + p_b, \quad (3.54)$$

and $q$ and $\pi(a)$ are the total heat flux and anisotropic stress in the frame defined by $u$:

$$q \equiv q_\nu + q_\gamma + (\rho_b + p_b)v_b \quad (3.55)$$

$$\pi(a) \equiv \pi_\nu(a) + \pi_\gamma(a). \quad (3.56)$$

Note that $\rho$ and $p$ are independent of the choice of $u$ to first-order, and that there is no CDM contribution to $q$ since we have chosen $u$ equal to the CDM velocity.

The propagation equations are

$$-u\cdot\dot{\mathcal{D}}\dot{\mathcal{E}}(a) - \theta\mathcal{E}(a) - \mathcal{I}_{\mathcal{B}}(a) = \tfrac{1}{12}\kappa[6(\rho+p)\sigma(a) + 3(a\cdot\widehat{\mathcal{D}}q + \bar{\mathcal{D}}(a\cdot q)) - 6u\cdot\dot{\mathcal{D}}\dot{\pi}(a) - 2\theta\pi(a) - 2H(a)\widehat{\mathcal{D}}\cdot q] \quad (3.57)$$

$$-u\cdot\dot{\mathcal{D}}\dot{\mathcal{B}}(a) - \theta\mathcal{B}(a) + \mathcal{I}_{\mathcal{E}}(a) = -\tfrac{1}{4}\kappa[iu\wedge\bar{\mathcal{D}}\wedge\pi(a) + \pi(iu\wedge a\wedge\bar{\mathcal{D}})] \quad (3.58)$$

$$u\cdot\dot{\mathcal{D}}\dot{\sigma}(a) + \tfrac{2}{3}\theta\sigma(a) + \mathcal{E}(a) + \tfrac{1}{2}\kappa\pi(a) = 0 \tag{3.59}$$

$$u\cdot\mathcal{D}\varpi + \tfrac{2}{3}\theta\varpi = 0, \tag{3.60}$$

where $\mathcal{I}_{\mathcal{B}}(a)$ is a trace-free, symmetric, spatial tensor defined by

$$\mathcal{I}_{\mathcal{B}}(a) \equiv \tfrac{1}{2}[iu\wedge\bar{\mathcal{D}}\wedge\mathcal{B}(a) + \mathcal{B}(iu\wedge a\wedge\bar{\mathcal{D}})], \tag{3.61}$$

where

$$\mathcal{B}(iu\wedge a\wedge\bar{\mathcal{D}}) \equiv H[H(b)\cdot\dot{\mathcal{D}}\dot{\mathcal{B}}(iu\wedge a\wedge\partial_b)], \tag{3.62}$$

and $\mathcal{I}_{\mathcal{E}}(a)$ is defined similarly. There is some redundancy in these equations. For example, equations (3.48), (3.51) and the integrability condition

$$\widehat{\mathcal{D}}\wedge\widehat{\mathcal{D}}\theta = -\varpi u\cdot\mathcal{D}\theta, \tag{3.63}$$

imply the constraint (3.49). Similarly, equation (3.48) along with the propagation equations (3.59) and (3.60) imply the propagation equation (3.58). It follows that $\mathcal{B}(a)$ may be eliminated from the full set of equations by making use of the constraint (3.48). This turns out to be a necessary step when the covariant first-order quantities are harmonically expanded for scalar, vector or tensor perturbations.

Some comments are in order regarding the equations presented in this section. The equations are both covariant and gauge-invariant (in the sense of being independent of any map between the lumpy universe and the FRW model). Gauge-invariance ensures that the gauge-problems that have plagued other approaches are not present, while covariance ensures that we are working with quantities which are straightforward to interpret physically. The equations describe scalar, vector and tensor perturbations in a unified manner, and are independent of any harmonic analysis into spatial modes. Furthermore, we have not had to specify the background FRW model yet (there is an implicit assumption that the universe is approximately FRW when the first-order, covariant variables are constructed).

## 3.5 Scalar Perturbations in a $K = 0$ Universe

In this section we reduce the general equations of the previous section to a set of equations for scalar-valued, first-order gauge-invariant variables describing the evolution of the density inhomogeneities and the CMB anisotropy for scalar perturbations in a universe which is approximately a $K = 0$ FRW model. These equations thus describe the standard CDM model in a covariant and gauge-invariant manner. The equations split into a set of algebraic relations (from the constraint equations) and propagation equations for scalar variables. The moment equations for the photon and neutrino distributions (for $l \geq 3$) are equivalent to those given elsewhere [37], where a Fourier expansion of the spatial variation, and a Legendre expansion of the angular dependence of the CMB anisotropy are made.

Scalar perturbations may be characterised in a covariant manner by demanding that the magnetic part of the Weyl tensor and the vorticity both vanish identically. The vanishing of the vorticity ensures that $u$ is a hypersurface orthogonal vector field. The perturbation equations place strong restrictions on the remaining non-zero variables, which may be satisfied by constructing the vector and tensor variables from spatial derivatives of the scalar (harmonic) eigenfunctions $Q_k$ of the generalised Helmholtz equation:

$$\widehat{\mathcal{D}}^2 Q_k = \tfrac{k^2}{S^2} Q_k, \tag{3.64}$$

which are constructed to satisfy

$$u\cdot\mathcal{D} Q_k = \mathrm{O}(1). \tag{3.65}$$

From the $Q_k$ we form a scalar valued tensor $Q_k^{(1)}(a)$:

$$Q_k^{(1)}(a) \equiv \tfrac{S}{k} a\cdot\widehat{\mathcal{D}} Q_k, \tag{3.66}$$

which has the properties

$$Q_k^{(1)}(u) = 0, \qquad u\cdot\dot{\mathcal{D}}\dot{Q}_k^{(1)}(a) = \mathrm{O}(1). \tag{3.67}$$

We then define scalar-valued tensors $Q_k^{(l)}(a_1,\ldots,a_l)$ by the recursion formula (for $l>1$)

$$Q_k^{(l)}(a_1,\ldots,a_l) = \tfrac{1}{l!}\tfrac{S}{k}\Big(a_1\cdot\bar{\mathcal{D}} Q_k^{(l-1)}(a_2,\ldots,a_l) - \tfrac{l-1}{2l-1} H(a_1)\cdot H(a_2) Q_k^{(l-1)}(\bar{\mathcal{D}}, a_3,\ldots,a_l) \\ + \text{perms}\Big). \tag{3.68}$$

These tensors satisfy the properties

$$\begin{aligned} Q_k^{(l)}(a_1,\ldots,a_j,\ldots,a_k,\ldots,a_l) &= Q_k^{(l)}(a_1,\ldots,a_k,\ldots,a_j,\ldots,a_l) \\ Q_k^{(l)}(u,a_2,\ldots,a_l) &= 0 \\ Q_k^{(l)}(\partial_b, b, a_3,\ldots,a_l) &= 0 \\ u\cdot\dot{\mathcal{D}}\dot{Q}_k^{(l)}(a_1,\ldots,a_l) &= \mathrm{O}(1), \end{aligned} \tag{3.69}$$

which are readily proved by induction. We shall also make use of the vector $Q_k^{(v)} \equiv \partial_a Q_k^{(1)}(a)$ and the vector-valued tensor $Q_k(a) \equiv \partial_b Q_k^{(2)}(a,b)$.

The definitions above are independent of the spatial curvature $K$ of the background model. However, the differential properties of the $Q_k^{(l)}(a_1,\ldots,a_l)$ are dependent on the value of $K$. Some of these properties are listed in the appendix to Bruni *et al.* [38] The only new result we require is

$$Q_k^{(l)}(\bar{\mathcal{D}}, a_2,\ldots,a_l) = \tfrac{l}{2l-1}\tfrac{k}{S} Q_k^{(l-1)}(a_2,\ldots,a_l), \tag{3.70}$$

which is valid for $K = 0$ only.

We separate out the spatial and temporal dependence of the covariant, first-order variables by expanding in the harmonics derived from the $Q_k$ in the following manner:

$$\mathcal{X}_i = \sum_k k\mathcal{X}_{ik}Q_k^{(v)}, \qquad \mathcal{Z} = \sum_k \tfrac{k^2}{S}\mathcal{Z}_k Q_k^{(v)} \tag{3.71}$$

$$\mathcal{E}(a) = \sum_k \left(\tfrac{k}{S}\right)^2 \Phi_k Q_k(a), \qquad \sigma(a) = \sum_k \left(\tfrac{k}{S}\right)\sigma_k Q_k(a) \tag{3.72}$$

$$q_\nu = \rho_\nu \sum_k q_{\nu k}Q_k^{(v)}, \qquad q_\gamma = \rho_\gamma \sum_k q_{\gamma k}Q_k^{(v)} \tag{3.73}$$

$$\pi_\nu(a) = \rho_\nu \sum_k \pi_{\nu k}Q_k(a), \qquad \pi_\gamma(a) = \rho_\gamma \sum_k \pi_{\gamma k}Q_k(a). \tag{3.74}$$

We also expand $v_b$ as

$$v_b = \sum_k v_k Q_k^{(v)}. \tag{3.75}$$

Finally, we assume that the higher-order moments of the (energy-integrated) neutrino and photon distribution functions may also be expanded in harmonics. By considering the zero-order form of the scalar harmonics $Q_k$, it is straightforward to show that this assumption is equivalent to that which is usually made [37] (the Fourier components of the distribution function are axisymmetric about the wavevector $\boldsymbol{k}$). The procedure we have adopted here, of expanding the angular dependence of the distribution function in symmetric, traceless, covariant spatial tensors, which are then expanded in the appropriate harmonic tensors, derived from the scalar harmonics, should be compared to the usual approach of Fourier expanding (for $K = 0$) all variables, and then performing a Legendre expansion of the angular dependence of the distribution function modes about the wavevector $\boldsymbol{k}$. Although ultimately equivalent, the approach taken here has the advantage that it goes over unchanged to the case of tensor perturbations (although it is now $Q_k^{(2)}(a_1, a_2)$ which satisfies the generalised Helmholtz equation). For later convenience, we expand the angular moments of the distribution function in the form

$$J_\nu^{(l)}(a_1, \ldots, a_l) = \rho_\nu \sum_k J_{\nu k}^{(l)} Q_k^{(l)}(a_1, \ldots, a_l) \tag{3.76}$$

$$J_\gamma^{(l)}(a_1, \ldots, a_l) = \rho_\gamma \sum_k J_{\gamma k}^{(l)} Q_k^{(l)}(a_1, \ldots, a_l). \tag{3.77}$$

It is now a simple matter to substitute the harmonic expansions of the covariant variables into the general equations of Section 3.4 (with $\mathcal{B}(a)$ and $\varpi$ set to zero), to obtain equations for the scalar expansion coefficients. We assume that the variations in baryon pressure $p_b$ due to entropy variations are insignificant compared to those due to variations in $\rho_b$. Accordingly, we write

$$\mathcal{D}p_b = c_s^2 \mathcal{D}\rho_b, \tag{3.78}$$

where $c_s$ is the adiabatic sound speed in the baryon/electron fluid (this is different from the sound speed in the coupled baryon/photon plasma).

With this assumption we obtain the following equations for scalar perturbations in a $K = 0$ universe: the spatial gradients of the densities give

$$\dot{\mathcal{X}}_{\nu k} = -\tfrac{4}{3}\tfrac{k}{S}\mathcal{Z}_k - \tfrac{k}{S}q_{\nu k} \tag{3.79}$$

$$\dot{\mathcal{X}}_{\gamma k} = -\tfrac{4}{3}\tfrac{k}{S}\mathcal{Z}_k - \tfrac{k}{S}q_{\gamma k} \tag{3.80}$$

$$\dot{\mathcal{X}}_{bk} = -\left(1 + \tfrac{p_b}{\rho_b}\right)\tfrac{k}{S}(\mathcal{Z}_k + v_k) + \left(\tfrac{p_b}{\rho_b} - c_s^2\right)\theta\mathcal{X}_{bk} \tag{3.81}$$

$$\dot{\mathcal{X}}_{ck} = -\tfrac{k}{S}\mathcal{Z}_k, \tag{3.82}$$

(an overdot on a scalar denotes $u\cdot\mathcal{D}$), the spatial gradient of $\theta$ gives

$$\tfrac{k}{S}\dot{\mathcal{Z}}_k + \tfrac{1}{3}\tfrac{k}{S}\theta\mathcal{Z}_k + \tfrac{1}{2}\kappa\Big(2\rho_\gamma\mathcal{X}_{\gamma k} + 2\rho_\nu\mathcal{X}_{\nu k} + (1+3c_s^2)\rho_b\mathcal{X}_{bk} + \rho_c\mathcal{X}_{ck}\Big) = 0, \tag{3.83}$$

the heat fluxes give

$$\dot{q}_{\nu k} + \tfrac{2}{3}\tfrac{k}{S}\pi_{\nu k} - \tfrac{1}{3}\tfrac{k}{S}\mathcal{X}_{\nu k} = 0 \tag{3.84}$$

$$\dot{q}_{\gamma k} + \tfrac{2}{3}\tfrac{k}{S}\pi_{\gamma k} - \tfrac{1}{3}\tfrac{k}{S}\mathcal{X}_{\gamma k} = \sigma_T n_e\left(\tfrac{4}{3}v_k - q_{\gamma k}\right) \tag{3.85}$$

$$\left(1 + \tfrac{p_b}{\rho_b}\right)\left(\dot{v}_k + \tfrac{1}{3}(1-3c_s^2)\theta v_k\right) - c_s^2\tfrac{k}{S}\mathcal{X}_{bk} = -\tfrac{\rho_\gamma}{\rho_b}\sigma_T n_e\left(\tfrac{4}{3}v_k - q_{\gamma k}\right), \tag{3.86}$$

the remaining moment equations are, for $l \geq 3$

$$\dot{\pi}_{\nu k} + \tfrac{3}{5}\tfrac{k}{S}J^{(3)}_{\nu k} - \tfrac{2}{5}\tfrac{k}{S}q_{\nu k} - \tfrac{8}{15}\tfrac{k}{S}\sigma_k = 0 \tag{3.87}$$

$$\dot{\pi}_{\gamma k} + \tfrac{3}{5}\tfrac{k}{S}J^{(3)}_{\gamma k} - \tfrac{2}{5}\tfrac{k}{S}q_{\gamma k} - \tfrac{8}{15}\tfrac{k}{S}\sigma_k = -\tfrac{9}{10}\sigma_T n_e\pi_{\gamma k} \tag{3.88}$$

$$\dot{J}^{(l)}_{\nu k} + \tfrac{k}{S}\left(\tfrac{l+1}{2l+1}J^{(l+1)}_{\nu k} - \tfrac{l}{2l+1}J^{(l-1)}_{\nu k}\right) = 0 \tag{3.89}$$

$$\dot{J}^{(l)}_{\gamma k} + \tfrac{k}{S}\left(\tfrac{l+1}{2l+1}J^{(l+1)}_{\gamma k} - \tfrac{l}{2l+1}J^{(l-1)}_{\gamma k}\right) = -\sigma_T n_e J^{(l)}_{\gamma k}, \tag{3.90}$$

the electric part of the Weyl tensor gives

$$\begin{aligned} -\left(\tfrac{k}{S}\right)^2\left(\dot{\Phi}_k + \tfrac{1}{3}\theta\Phi_k\right) = {}& \tfrac{1}{2}\kappa\tfrac{k}{S}\Big((\rho+p)\sigma_k + \rho_\nu q_{\nu k} + \rho_\gamma q_{\gamma k} + (\rho_b + p_b)v_k\Big) \\ & + \tfrac{1}{2}\kappa\theta(\rho_\nu\pi_{\nu k} + \rho_\gamma\pi_{\gamma k}) - \tfrac{1}{2}\kappa(\rho_\nu\dot{\pi}_{\nu k} + \rho_\gamma\dot{\pi}_{\gamma k}), \end{aligned} \tag{3.91}$$

and the shear tensor gives

$$\tfrac{k}{S}\left(\dot{\sigma}_k + \tfrac{1}{3}\theta\sigma_k\right) + \left(\tfrac{k}{S}\right)^2\Phi_k + \tfrac{1}{2}\kappa(\rho_\nu\pi_{\nu k} + \rho_\gamma\pi_{\gamma k}) = 0. \tag{3.92}$$

Finally, the harmonic expansions of the constraint equations give

$$\begin{aligned} 2\left(\tfrac{k}{S}\right)^3\Phi_k - \kappa\tfrac{k}{S}\Big(\rho_\nu(\mathcal{X}_{\nu k} + \pi_{\nu k}) + \rho_\gamma(\mathcal{X}_{\gamma k} + \pi_{\gamma k}) + \rho_c\mathcal{X}_{ck} + \rho_b\mathcal{X}_{bk}\Big) & \\ - \kappa\theta\Big(\rho_\nu q_{\nu k} + \rho_\gamma q_{\gamma k} + (\rho_b + p_b)v_k\Big) = 0, & \end{aligned} \tag{3.93}$$

and

$$\tfrac{2}{3}\left(\tfrac{k}{S}\right)^2(\mathcal{Z}_k-\sigma_k)+\kappa\Big(\rho_\nu q_{\nu k}+\rho_\gamma q_{\gamma k}+(\rho_b+p_b)v_k\Big)=0. \tag{3.94}$$

It is straightforward to show that these constraint equations are preserved by the propagation equations.

These gauge-invariant equations should be compared to the (gauge-dependent) equations used in most calculations of the CMB anisotropy (see, for example Ma and Bertshinger [37]). Note that the equations for $J_{\gamma k}^{(l)}$ and $J_{\nu k}^{(k)}$ for $l \geq 3$ are equivalent to those usually found in the literature since the moments of the distribution function in such gauge-dependent calculations are actually gauge-invariant for $l \geq 3$.

## 3.6 The CMB temperature anisotropy

The equations of the previous section may be solved numerically for appropriate initial conditions [2] to give the $J_{\gamma k}^{(l)}$ at the present epoch. These moments fully describe the CMB temperature anisotropy. Denote the full sky average of the CMB temperature by $T_0$, and the (gauge-invariant) temperature difference from the mean along a spatial direction $e$ by $\delta T(e)$. Then we have [23]

$$\frac{(T_0+\delta T(e))^4}{T_0^4}=\frac{\sum_{l=0}^{\infty}\int dE\,E^3F_\gamma^{(l)}(e,\ldots,e)}{\int dE\,E^3F_\gamma^{(0)}}, \tag{3.95}$$

so that, to first-order,

$$4\frac{\delta T(e)}{T_0}=\frac{4\pi}{\rho_\gamma}\sum_{l=1}^{\infty}\int dE\,E^3F_\gamma^{(l)}(e,\ldots,e). \tag{3.96}$$

Recalling the definition (3.25), we find

$$\frac{\delta T(e)}{T_0}=-\frac{3}{4}\frac{q_\gamma\cdot e}{\rho_\gamma}+\frac{15}{8}\frac{\pi_\gamma(e)\cdot e}{\rho_\gamma}+\frac{1}{4\rho_\gamma}\sum_{l=3}^{\infty}\frac{(2l+1)(2l)!}{(-2)^l(l!)^2}J_\gamma^{(l)}(e,\ldots,e). \tag{3.97}$$

The right-hand side of (3.97) is the covariant harmonic expansion of the temperature anisotropy. The temperature anisotropy may be expanded in the more familiar spherical harmonics by introducing a covariant triad, orthogonal to $u$, at the observation point. We now have all the ingredients needed to calculate the CMB anisotropy for scalar perturbations in a flat CDM model, for given initial conditions. The numerical calculation of the anisotropy for such a model are underway, and the results will be reported elsewhere [2]. Our results should confirm the calculations of other groups, once the gauge-invariant information is extracted from their results (the CMB power spectrum, $C_l$, for $l > 0$).

## 3.7 Large Scale Anisotropies

On large angular scales, the dominant contributions to the anisotropy may be extracted analytically without recourse to a full numerical treatment. In this section, we show how this can be done within the framework developed above. Our gauge-invariant treatment of large-scale anisotropies is complementary to the discussion by Ellis and Dunsby [32] elsewhere in this volume, who use a two-fluid model for the matter and radiation after recombination, rather than the kinetic theory approach adopted here.

For semi-analytic work, it is convenient to use the Boltzmann equation in the form of equation (3.20). Defining

$$\delta_T(e) \equiv \frac{\delta T(e)}{T_0}, \tag{3.98}$$

where $\delta T(e)$ is the gauge-invariant temperature fluctuation from the mean (see equation (3.95)), we can write (3.20) in the form

$$\begin{aligned}\delta_T(e)' + \sigma_T n_e \delta_T(e) = e\cdot\sigma(e) + e\cdot w - \frac{1}{3}\theta(1+4\delta_T(e)) - \frac{\rho_\gamma'}{4\rho_\gamma}(1+4\delta_T(e))\\ - \sigma_T n_e\left(e\cdot v_b - \frac{3}{16}\frac{e\cdot\pi_\gamma(e)}{\rho_\gamma}\right),\end{aligned} \tag{3.99}$$

where a prime denotes differentiation with respect to the parameter $\lambda$ along the geodesic, where, for this section, we take $(u+e)\cdot\mathcal{D}\lambda = 1$. Equation (3.99) is correct to first-order. Note that in writing (3.99) we have not made a physical choice for the velocity $u$ yet, so we allow the possibility that $w$ does not vanish. (In the previous section we took $u$ to coincide with the velocity of the CDM, but we relax that restriction here.) Writing the Boltzmann equation in the form (3.99) is useful since the equation may be integrated along the geodesic to determine the temperature anisotropy along the given direction on the sky. Before doing this, it is convenient to eliminate $\theta$ from (3.99). In this section we shall consider a model with interacting baryons and radiation only, so that we describe the same situation as Ellis and Dunsby elsewhere in this volume [32] (see also Dunsby [39]). In this case, it is convenient to define $u$ to be equal to the velocity of the baryons, so that $v_b = 0$. Using (3.32), and neglecting baryon pressure, we find

$$\theta = -\frac{\rho_b'}{\rho_b} + \frac{e\cdot\widehat{\mathcal{D}}\rho_b}{\rho_b}, \tag{3.100}$$

and using (3.33), we may write the acceleration as

$$w = \sigma_T n_e \frac{q_\gamma}{\rho_b}, \tag{3.101}$$

which shows that after recombination, the acceleration is negligible, particularly in a universe which is matter dominated at recombination. Substituting into (3.99), we find

$$\delta_T(e)' + \sigma_T n_e \delta_T(e) = e\cdot\sigma(e) - \frac{1}{3}\frac{e\cdot\widehat{\mathcal{D}}\rho_b}{\rho_b} + \frac{\rho_b'}{3\rho_b} - \frac{\rho_\gamma'}{4\rho_\gamma} + \sigma_T n_e \left(\frac{e\cdot q_\gamma}{\rho_b} + \frac{3}{16}\frac{e\cdot\pi_\gamma(e)}{\rho_\gamma}\right). \tag{3.102}$$

This equation may be integrated up the null geodesic from some point in the distant past (where $\lambda = \lambda_i$) to the reception point $R$ (where $\lambda = \lambda_R$). Introducing the optical depth $\kappa(\lambda)$ along the line of sight, defined by

$$\kappa(\lambda) = \int_\lambda^{\lambda_R} n_e \sigma_T \, d\lambda, \tag{3.103}$$

we find that (3.102) integrates to give

$$\Big(\delta_T(e)\Big)_R = \int_{\lambda_i}^{\lambda_R} -\kappa' e^{-\kappa}\left[\frac{e\cdot q_\gamma}{\rho_b} + \frac{3}{16}\frac{e\cdot\pi_\gamma(e)}{\rho_\gamma} - \frac{1}{3}\ln\rho_b + \frac{1}{4}\ln\rho_\gamma\right] + e^{-\kappa}\left[e\cdot\sigma(e) - \frac{1}{3}\frac{e\cdot\widehat{\mathcal{D}}\rho_b}{\rho_b}\right] d\lambda, \tag{3.104}$$

where we have integrated by parts, we have assumed that terms evaluated at $\lambda_i$ are negligible due to the large optical depth, and we have neglected a direction-independent (monopole) term which must be cancelled by other terms in the integral. The notation $(A)_R$ denotes the quantity $A$ evaluated at the point $R$. An expression similar to (3.104) forms the starting point of the 'line of sight integration approach' to calculating CMB anisotropies [40], but note that (3.104) is true for scalar, vector and tensor modes in almost FRW universes with any value of $K$, and has not assumed any Fourier decomposition. Furthermore, the expression only involves physically-defined gauge-invariant variables, such as the fractional temperature fluctuation from the mean $\delta_T(e)$.

The quantity $-\kappa' e^{-\kappa}$ defines the visibility function, which is the probability density that a photon was last scattered at $\lambda$. The visibility function peaks at a redshift $z \simeq 1100$ and has a dispersion $\simeq 70$ in redshift [41]. It follows that on angular scales larger than $8'\Omega^{-1/2}$, the visibility function may be approximated by a $\delta$-function, whose support defines the last scattering surface. With this approximation, $e^{-\kappa} = 0$ before last scattering (tight-coupling), and $e^{-\kappa} = 1$ after last scattering (free-streaming), so that equation (3.104) reduces to

$$\Big(\delta_T(e)\Big)_R = \left(\frac{e\cdot q_\gamma}{\rho_b} + \frac{3}{16}\frac{e\cdot\pi_\gamma(e)}{\rho_\gamma} - \frac{1}{3}\ln\rho_b + \frac{1}{4}\ln\rho_\gamma\right)_A + \int_{\lambda_A}^{\lambda_R}\left[e\cdot\sigma(e) - \frac{1}{3}\frac{e\cdot\widehat{\mathcal{D}}\rho_b}{\rho_b}\right] d\lambda, \tag{3.105}$$

where $A$ is the point where the geodesic intersects the last scattering surface. Taking the difference between two directions on the sky, we find the following expression for the gauge-invariant temperature difference $\Delta T$

$$\left(\frac{\Delta T}{T_0}\right)_R = \frac{1}{4}\Delta(\ln\rho_\gamma)_E - \frac{1}{3}\Delta(\ln\rho_b)_E + \frac{\Delta(e\cdot q_\gamma)_E}{\rho_b} + \frac{3}{16}\frac{\Delta(e\cdot\pi_\gamma(e))_E}{\rho_\gamma} + \Delta\int\left[e\cdot\sigma(e) - \frac{1}{3}\frac{e\cdot\widehat{\mathcal{D}}\rho_b}{\rho_b}\right]d\lambda, \quad (3.106)$$

where, for example,

$$\Delta(\ln\rho_\gamma)_E \equiv (\ln\rho_\gamma)_A - (\ln\rho_\gamma)_B, \quad (3.107)$$

with $B$ the point of intersection of the other null geodesic with the last scattering surface, and

$$\Delta\int(\;)\,d\lambda \equiv \int_{\lambda_A}^{\lambda_R}(\;)\,d\lambda - \int_{\lambda_B}^{\lambda_R}(\;)\,d\lambda. \quad (3.108)$$

The first term on the right-hand side of (3.106) is just the fractional difference in radiation temperature between the points of emission $A$ and $B$ on the last scattering surface, while the second term describes the effects of inhomogeneity in the baryon density. It is convenient to write the sum of these two terms as a line integral in the last scattering surface. Denoting by $dx_{\rm cov}$ the covariant element of length for an arbitrary path connecting $A$ and $B$, we have

$$\begin{aligned}\frac{1}{4}\Delta(\ln\rho_\gamma)_E - \frac{1}{3}\Delta(\ln\rho_b)_E &= \int_B^A\left[\frac{\mathcal{D}\rho_\gamma}{4\rho_\gamma} - \frac{\mathcal{D}\rho_b}{3\rho_b}\right]\cdot dx_{\rm cov} \\ &= \int_B^A\left[\left(\frac{\widehat{\mathcal{D}}\rho_\gamma}{4\rho_\gamma} - \frac{\widehat{\mathcal{D}}\rho_b}{3\rho_b}\right) + u\left(\frac{u\cdot\mathcal{D}\rho_\gamma}{4\rho_\gamma} - \frac{u\cdot\mathcal{D}\rho_b}{3\rho_b}\right)\right]\cdot dx_{\rm cov} \\ &= \int_B^A[\mathcal{S}_{\gamma b} - \tfrac{1}{3}u\widehat{\mathcal{D}}\cdot v_\gamma]\cdot dx_{\rm cov}, \end{aligned} \quad (3.109)$$

where

$$\mathcal{S}_{\gamma b} \equiv \frac{\widehat{\mathcal{D}}\rho_\gamma}{4\rho_\gamma} - \frac{\widehat{\mathcal{D}}\rho_b}{3\rho_b}, \qquad v_\gamma \equiv \frac{3q_\gamma}{4\rho_\gamma}. \quad (3.110)$$

The covariant vector $\mathcal{S}_{\gamma b}$ is a covariant measure of the entropy perturbations (see, for example, Bruni *et al.* [38]), and $v_\gamma$ is the effective radiation velocity (relative to the baryon frame). If we take the integration path to lie entirely in the last scattering surface, $u\cdot dx_{\rm cov} = \mathrm{O}(1)$, so that (3.109) reduces to

$$\frac{1}{4}\Delta(\ln\rho_\gamma)_E - \frac{1}{3}\Delta(\ln\rho_b)_E = \int_B^A \mathcal{S}_{\gamma b}\cdot dx_{\rm cov}, \quad (3.111)$$

which shows that the contribution of the first two terms in (3.106) depends only on the entropy perturbations on the last scattering surface. In particular, for perturbations which are exactly adiabatic at last scattering, there is no contribution from these terms. The third term in (3.106) is a Doppler term which is negligible for a universe which is matter dominated at recombination, and the fourth arises from viscous effects in the photons. Before recombination, both $q_\gamma$ and $\pi_\gamma$ are first-order in the photon mean free time $(n_e \sigma_T)^{-1}$ (recall that we have defined $u$ to coincide with the baryon velocity), so that in the tight-coupling/instantaneous recombination approximation these terms may be ignored. It follows that for perturbations which are exactly adiabatic at last scattering, only the integral term on the right-hand side of (3.106) remains.

An expression similar to (3.106) is derived elsewhere in this volume by Ellis and Dunsby [32], although they used a two fluid approach rather than the kinetic theory approach adopted here. To make contact with their work, we consider a universe which is already matter dominated at recombination. In this case, the baryon velocity coincides with the velocity of the energy frame (the frame in which the heat flux vanishes) which was used by Ellis and Dunsby, and we may replace $\rho_b$ by the total density $\rho$ in the integral in (3.105). Furthermore, we ignore the terms involving $q_\gamma$ and $\pi_\gamma$, as discussed above, to obtain

$$\Big(\delta_T(e)\Big)_R = \frac{1}{4}(\ln \rho_\gamma)_A - \frac{1}{3}(\ln \rho_b)_A + \int_{\lambda_A}^{\lambda_R} \left[ e\cdot\sigma(e) - \frac{1}{3}\frac{e\cdot\widehat{\mathcal{D}}\rho}{\rho}\right] d\lambda. \tag{3.112}$$

For ease of comparison we also give the corresponding expression from Ellis and Dunsby [32] in our notation:

$$\Big(\delta_T(e)\Big)_R = \mathcal{A} + \int_{\lambda_A}^{\lambda_R} \left[ e\cdot\sigma(e) - \frac{1}{3}\frac{e\cdot\widehat{\mathcal{D}}\rho}{\rho}\right] d\lambda, \tag{3.113}$$

where

$$\mathcal{A} \equiv -\int_{\lambda_A}^{\lambda_R} e\cdot(\mathcal{S}_{\gamma b} - \tfrac{1}{3}\widehat{\mathcal{D}}\cdot v_\gamma)\, d\lambda. \tag{3.114}$$

By making use of the rearrangement in (3.109), and noting that $dx_{\text{cov}} = (e+u)d\lambda$ along the geodesic, we find that $\mathcal{A}$ reduces to the first two terms on the right-hand side of (3.112) (after subtraction of a monopole term). It follows that the expression for the temperature anisotropy in Ellis and Dunsby is in agreement with that derived here, under the assumptions outlined above. Note that the part of $\mathcal{A}$ which is significant observationally is determined only by the entropy perturbation at last scattering, and so this term will vanish for perturbations which are adiabatic at last scattering, even though the perturbations do not remain adiabatic after decoupling.

### 3.8 Conclusion

The gauge-invariant calculation of CMB anisotropies may be performed in a fully covariant manner with the methods outlined in this section. The equations of Section 3.4 are a complete description of the evolution of perturbations in a CDM universe. They include all types of perturbation (scalar, vector and tensor) implicitly and are independent of the curvature of the background FRW model. We have demonstrated how the equation set may be reduced to scalar equations for the case of scalar perturbations in a $K = 0$ universe. The extension to $K \neq 0$ and vector or tensor modes is straightforward. The results of a numerical solution of the equations for scalar perturbations will be given elsewhere [2]. We expect that these calculations will confirm the result of other groups who have made their calculations by working carefully in specific gauges.

# 4 CMB Anisotropies in Global Defect Theories

Our final topic is a review of the results of some recent calculations by Pen, Seljak and Turok [6, 7] of the power spectra in global defect theories. Unlike the calculation of CMB anisotropies in inflationary models, the calculation for defect theories has been plagued by computational difficulties. These difficulties arise from the continual generation of perturbations by the causal source (the defects). One must solve for the non-linear evolution of the source, as well as the linearised response of the Einstein/fluid/Boltzmann equations to the causal sources.

Recently, an efficient technique has been developed for the calculation of power spectra in defect theories [6], which makes use of the unequal time correlator of the defect stress-energy tensor. This technique has provided the first accurate computations of the CMB and matter power spectra in global defect theories. The results of the calculation for the CMB power spectra (the $C_l$) are shown in Figure 4, for global defects comprised of strings, monopoles, textures and non-topological textures. The contributions from scalar, vector and tensor modes are shown separately, as well as their total. The most striking feature of the figure is the large contribution from vector modes to the power on large angular scales. The vector modes dominate the total signal for $l \lesssim 100$, at which point they are suppressed by the horizon size at recombination. Vector modes are insignificant in inflationary models since the vorticity decays as the universe expands to conserve angular momentum. However, in defect theories, the defects are a continual source of vorticity giving rise to a significant vector component to the CMB power spectrum. In addition, the continual (causal) sourcing of the fluid perturbations leads to decoherence, where scalar modes of a given scale (inside the sound horizon) oscillate with different phases in different parts of the universe, compared to the coherent oscillations which occur in inflationary models. This decoherence manifests itself as the suppression of the secondary Doppler peaks in Figure 4.

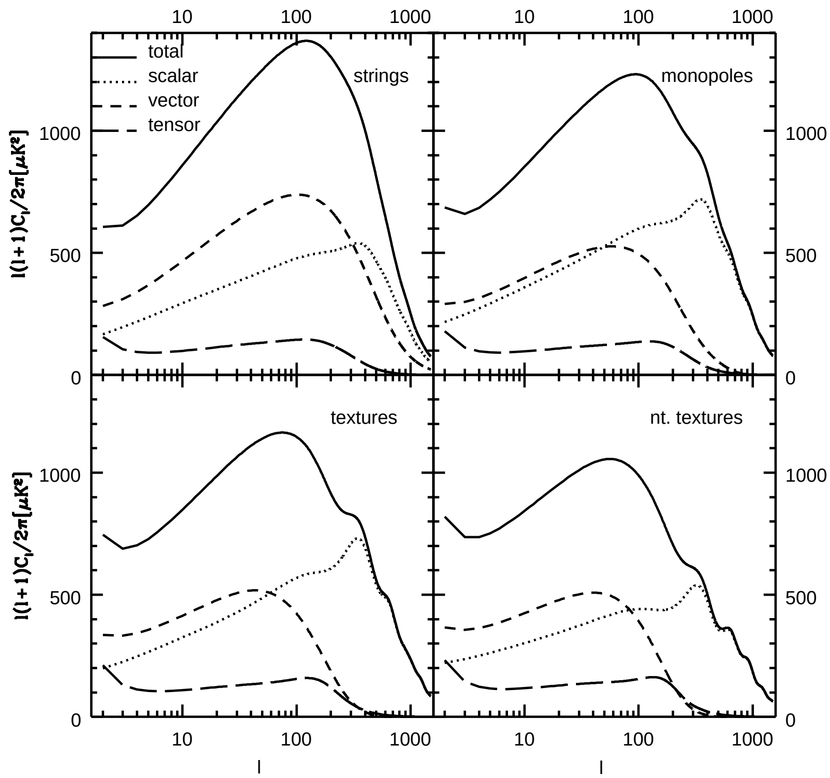


Figure 4: The contributions to the CMB power spectrum from scalar, vector and tensor components in global defect models with strings, monopoles, textures and non-topological textures. Reproduced with permission from Pen, Seljak and Turok (1997).

The significant contribution of the vector modes to the large scale power in the CMB spectrum leads to a suppression in the total signal for $l \geq 100$ compared to inflationary models, when both are normalised at large angular scales. In Figure 5 we show a comparison of the predictions of global defect theories, with COBE normalisation at $l = 10$, to the current generation of CMB measurements. The effect of vector modes and decoherence is to leave the predictions of the defect models systematically below the current degree-scale data. A similar effect occurs with the matter power spectrum, $P(k)$. The matter power spectrum calculated in global defect theories is compared with the predictions of a standard cold dark matter model (sCDM, with $H_0 = 50\mathrm{kms}^{-1}\,\mathrm{Mpc}^{-1}$ and a scale-invariant spectrum of initial perturbations) in Figure 6. The defect calculations and the sCDM calculations are normalised to COBE, and are compared with the matter power spectrum inferred from the galaxy distribution. The large contribution of vector modes to the CMB power spectrum in the range of the COBE normalisation, suppresses the scalar component compared to

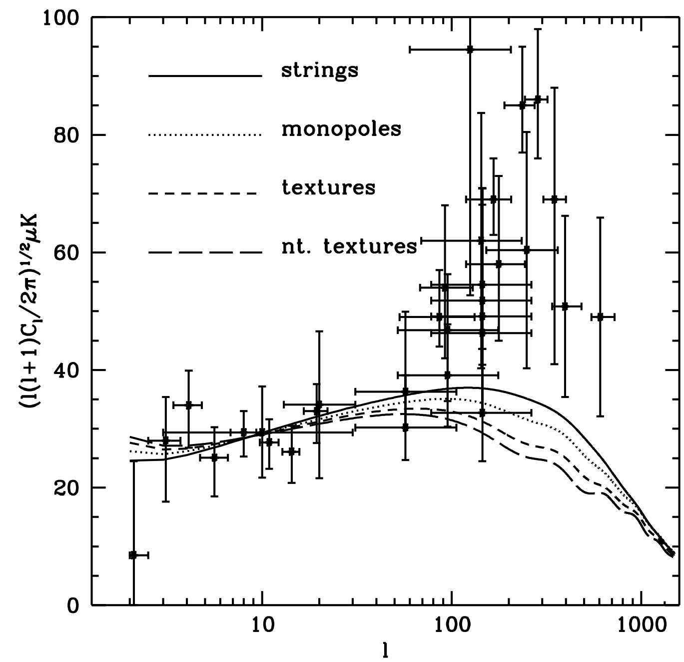


Figure 5: A comparison of the CMB power spectra calculated in defect models with the current observations. The calculated curves are normalised to COBE at $l = 10$. Reproduced with permission from Pen, Seljak and Turok (1997).

sCDM. The COBE normalised defect models predict too little power in the matter spectrum on larger scales, and too much power on smaller scales. As is well known, the sCDM model fares much better on large scales, but also predicts too much small scale power.

To conclude, current degree-scale observations of the CMB power spectrum, and the matter power spectrum inferred from galaxy clustering, do not favour global defect theories of structure formation, on the basis of the first accurate computations of the spectra in these theories. The significant contribution to large scale power from vector modes, which occurs in all defect models, leads to systematically low (COBE normalised) predictions for degree-scale CMB observations, and for the matter power spectrum.

# Acknowledgements

We thank U. Pen, U. Seljak and N. Turok for permission to reproduce Figures 4–6, and G. Ellis and P. Dunsby for allowing us access to the preprint of their contribution to this volume.

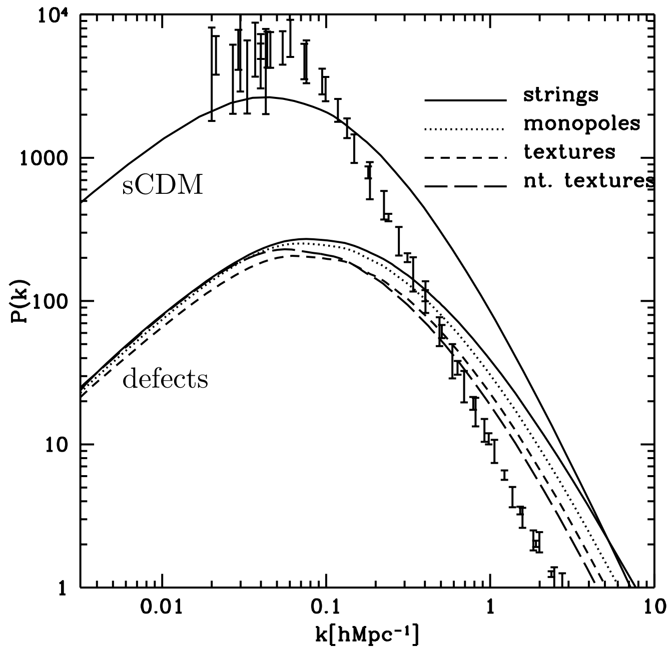


Figure 6: The matter power spectra computed in COBE normalised defect theories (lower curves) compared with the predictions of the standard CDM (sCDM) model (upper curve) for the same normalisation. The data points show the power spectrum inferred from galaxy distributions. Reproduced with permission from Pen, Seljak and Turok (1997).

# References


[1] A.D. Challinor and A.N. Lasenby. Relativistic corrections to the Sunyaev-Zel'dovich effect. To appear: *ApJ*, 1997.

[2] A.D. Challinor and A.N. Lasenby. Covariant and gauge-invariant calculations of CMB anisotropies. In preparation, 1997.

[3] G.F.R. Ellis and M. Bruni. Covariant and gauge-invariant approach to cosmological density fluctuations. *Phys. Rev. D*, 40:1804, 1989.

[4] G.F.R. Ellis, J. Hwang, and M. Bruni. Covariant and gauge-independent perfect-fluid Robertson-Walker perturbations. *Phys. Rev. D*, 40:1819, 1989.

[5] J.M. Bardeen. Gauge-invariant cosmological perturbations. *Phys. Rev. D*, 22:1882, 1980.

[6] U. Pen, U. Seljak, and N. Turok. Power spectra in global defect theories of cosmic structure formation. *Phys. Rev. Lett.*, 79:1611, 1997.

[7] U. Seljak, U. Pen, and N. Turok. Polarisation of the microwave background in defect models. *Phys. Rev. Lett.*, 79:1615, 1997.

[8] A.S. Kompaneets. The establishment of thermal equilibrium between quanta and electrons. *Soviet Physics JETP*, 4:730, 1957.

[9] E.L. Wright. Distortion of the microwave background by a hot intergalactic medium. *ApJ*, 232:348, 1979.

[10] R. Fabbri. Spectrum of the Sunyaev-Zel'dovich effect for high electron temperatures. *Astrophysics and Space Science*, 77:529, 1981.

[11] G.B. Taylor and E.L. Wright. Effects of a hot intergalactic medium. *ApJ*, 339:619, 1989.

[12] A. Loeb, C.F. McKee, and O. Lahav. Unsaturated Comptonization of isotropic photon spectra by relativistic electrons. *ApJ*, 374:44, 1991.

[13] Y. Rephaeli. Cosmic microwave background Comptonization by hot intracluster gas. *ApJ*, 445:33, 1995.

[14] A. Stebbins. The CMBR spectrum. To appear in: *Proceedings of the NATO ASI "The Cosmic Microwave Background", Strasbourg*, 1996.

[15] A. Stebbins. Extensions to the Kompaneets equation and the Sunyaev-Zel'dovich distortion. Submitted: *ApJ*, 1997.

[16] A.N. Lasenby and M.E. Jones. Constraints on the Hubble constant using the CMB. In N. Livio, M. Donahue, and N, Panagia, editors, *The Extragalactic Distance Scale*, page 76. Cambridge University Press, 1997.

[17] R. Saunders. Sunyaev-Zel'dovich observations with the Ryle telescope. In F.R. Bouchet, R. Gispert, B. Guiderdoni, and J. Tran Thanh Van, editors, *Microwave Background Anisotropies, 16th Moriond Astrophysics Meeting*, page 377. Editions Frontieres, 1996.

[18] Y. Rephaeli and D. Yankovitch. Relativistic corrections in the determination of $H_0$ from X-ray and Sunyaev-Zel'dovich measurements. *ApJ*, 481:L55, 1997.

[19] J.R. Buchler and W.R. Yeuh. Compton scattering opacities in a partially degenerate electron plasma at high temperatures. *ApJ*, 210:440, 1976.

[20] V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii. *Quantum Electrodynamics: Landau and Lifshitz Course of Theoretical Physics*, volume 4. Pergamon Press plc, Oxford, second edition, 1982.

[21] S.F. Gull and A. Garrett. In preparation, 1997.

[22] E.G. Corman. Relativistic Compton energy exchange. *Phys. Rev. D*, 1:2734, 1970.

[23] G.F.R. Ellis. The covariant and gauge-invariant approach to perturbations in cosmology. In N. Sánchez and A. Zichichi, editors, *Current Topics in Astrofundamental Physics: The Early Universe*, page 1. Kluwer Academic, Dordrecht, 1995.

[24] G.F.R. Ellis. The covariant approach to understanding anisotropy and inhomogeneity in cosmology. In N. Sánchez and A. Zichichi, editors, *Current Topics in Astrofundamental Physics*, page 3. World Scientific, Singapore, 1997.

[25] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Astrophysical and cosmological consequences of a gauge theory of gravity. In N. Sánchez and A. Zichichi, editors, *Advances in Astrofundamental Physics, Erice 1994*, page 359. World Scientific, Singapore, 1995.

[26] A.N. Lasenby, C.J.L. Doran, Y. Dabrowski, and A.D. Challinor. Rotating astrophysical systems and a gauge theory approach to gravity. In N. Sánchez and A. Zichichi, editors, *Current Topics in Astrofundamental Physics, Erice 1996*, page 380. World Scientific, Singapore, 1997.

[27] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Gravity, gauge theories and geometric algebra. To appear: *Phil. Trans. R. Soc. London. A*, 1997.

[28] C.J.L. Doran, A.N. Lasenby, A.D. Challinor, and S.F. Gull. Effects of spin-torsion in gauge theory gravity. To appear: *J. Math. Phys.*, 1997.

[29] A.D. Challinor, A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Massive non-ghost solutions for the Dirac field coupled self-consistently to gravity. To appear: *Gen. Rel. Grav.*, 1997.

[30] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus*. Reidel, Dordrecht, 1984.

[31] D. Hestenes. *Space-Time Algebra*. Gordon and Breach, New York, 1966.

[32] G.F.R. Ellis and P.K.S. Dunsby. Covariant analysis of dynamics and of CBR anisotropy. To appear in: N. Sánchez, editor, *Current Topics in Astrofundamental Physics, Erice 1997*. Kluwer Academic, Dordrecht, 1998.

[33] W.R. Stoeger, R. Maartens, and G.F.R. Ellis. Proving almost-homogeneity of the universe: an almost Ehlers-Geren-Sachs theorem. *ApJ*, 443:1, 1995.

[34] R. Maartens, G.F.R. Ellis, and W.R. Stoeger. Limits on anisotropy and inhomogeneity from the cosmic background radiation. *Phys. Rev. D*, 51:1525, 1995.

[35] G.F.R. Ellis, D.R. Matravers, and R. Treciokas. Anisotropic solutions of the Einstein-Boltzmann equations: I. General formalism. *Annals of Physics*, 150:455, 1983.

[36] W. Hu, and D. Scott, N. Sugiyama, and M. White. The effect of physical assumptions on the calculation of microwave background anisotropies. *Phys. Rev. D*, 52:5498, 1995.

[37] C.-P. Ma and E. Bertshinger. Cosmological perturbation theory in the synchronous and conformal Newtonian gauges. *ApJ*, 455:7, 1995.

[38] M. Bruni, P.K.S. Dunsby, and G.F.R. Ellis. Cosmological perturbations and the meaning of gauge-invariant variables. *ApJ*, 395:34, 1992.

[39] P.K.S. Dunsby. A fully covariant description of CMBR anisotropies. To appear: *Class. Quantum. Grav.*, 1997.

[40] U. Seljak and M. Zaldarriaga. A line of sight integration approach to cosmic microwave background anisotropies. *ApJ*, 469:437, 1996.

[41] B.J.T. Jones and R.F.G. Wyse. The ionisation of the primeval plasma at the time of recombination. *Astron. Astrophys.*, 149:144, 1985.